Introduction

1. Signals
    1. deterministic signal
    2. Non-deterministic signal / Random signal

1. deterministic signal :-

A signal whose future values can be predicted (or) estimated is called deterministic signal.

Eg:- Sinx

we can define a mathematical relation for a deterministic signal which is shown in the above example.

2. Non-deterministic signal / Random signal :-

If the future values of a signal can't be predicted than it is called a random signal.

2. Sample Space :-
    1. discrete Sample space
    2. continuous sample space

Sample space :-

Set of all possible outcomes of an experiment

Eg:- Throwing a die

$S = \{1,2,3,4,5,6\}$

① discrete sample space :-

A sample space is said to be discrete and finite, of the set in the sample space have finite number of elements

Eg :- The set $S=\{1,2,3,4,5\}$ is a finite set

If the sets in the sample space have infinite number of elements, then the sample space is discrete and Infinite.

② continuous sample space :-

If the sample space contains an Infinite number of elements with continuous values with in a given range, then it is called a continuous sample space.

for example, the set of numbers from 0 to 10. The elements are infinite and continuous

3. event :-

Set of possible outcomes of an experiment

Eg :- throwing a die
Getting a prime number
$A = \{2, 3, 5\}$

Types of events

① certain event :-

If the probability of occuring of an event then it is certain event (or) sure event

Eg $\boxed{P(A) = 1}$

② Impossible event :-

If the probability of occuring of an event is 0 then it is Impossible event

$\boxed{P(A) = 0}$

$\boxed{0 \leq P(A) \leq 1}$

Note:-

The probability of an event is always lies between 0 and 1

③ mutually exclusive event:-

If two events of an experiment have no common outcomes than the two events are said to be mutually exclusive event

Eg:- Throwing a die

$S = \{1, 2, 3, 4, 5, 6\}$

Event A = Getting a prime number

$A = \{2, 3, 5\}$

Event B = Getting a number greater than 5

$B = \{6\}$

$\boxed{P(A \cap B) = 0}$

④ Independent event:-

consider two events A and B in a sample space S having Non-zero probabilities.

If the probability of occurance of one of the event is not affected by the occurence of the other event, then the events are said to be "statically independent" events

Mathematically, two events are statically independent if and only if

$\boxed{P(A \cap B) = P(A)P(B) \text{ for } P(A) \neq 0 \text{ and } P(B) \neq 0}$

Eg:- consider an experiment of tossing a coin, throwing a die

Event = A = Getting a Head on tossing a coin

Event = B = Getting a number greater than 3 on throwing a die

$A = \{H\} \Rightarrow P(A) = \frac{1}{2}$

$B = \{4,5,6\} \Rightarrow P(B) = \frac{3}{6}$

(5) exhaustive events:-

All possible events in a Sample space are called exhaustive events.

for example, consider an experiment of throwing a die. The exhaustive events are six when two dice are thrown, the exhaustive events are 36.

(6) equally likely events:-

On a given experiment, if one of the events in a sample space does not depend on another event, then the two events are called equally likely events.

for example, By throwing an unbiased die, all the faces shown are equally likely events. Also, while tossing an unbiased coin, getting "Heads" and "tails" are equally likely events

4. Probability :-

Probability is the ratio of No. of favourable outcomes to the total No. of outcomes of an experiment

$$\text{Probability} = \frac{\text{No. of favourable Outcomes}}{\text{Total No. of outcomes}}$$

① Relative frequency approach :-

By relative frequency approach probability of an event (A) is the ratio of number of times the event (A) occured to the number of times the experiment is conducted

Eg :- $P(A) = \lim_{n \to \infty} \frac{n(A)}{n}$

② Axiomatic approach / classical approach :-

The probability of an event (A) is the ratio of number of favourable outcomes to the total number of outcomes

$$P(A) = \frac{\text{No. of favourable outcomes}}{\text{Total No. of outcomes}}$$

Eg :- Throwing a die

$S = \{1, 2, 3, 4, 5, 6\}$

A = getting an even number

$\Rightarrow 3/6 \Rightarrow 1/2 \Rightarrow 0.5$

Problems

(1) When a Single Coin tossed, what is the Probability of getting one head.

Sol

Sample Space $S = \{H, T\}$

Event A = Getting One head

$$P(A) = \frac{\text{No. of fovourable outcomes}}{\text{Total No. of outcomes}}$$

$$= \frac{1}{2}$$

$$P(A) = 0.5$$

(2) When two coins tossed. Find the Probability of A = Getting atleast one head, B = Getting two heads.

Sol

$$S = \{HH, HT, TH, TT\}$$

Event A = Getting atleast one head

$$P(A) = \frac{3}{4}$$

Event B = Getting two heads

$$P(B) = \frac{1}{4}$$

(3) When 3 coins tossed, What is the Probability of getting more than one head.

Sol

$$S = \left\{\begin{matrix} HHH, & HHT, & HTH, & HTT \\ THH, & THT, & TTH, & TTT \end{matrix}\right\}$$

Event A = Getting more than one head

$A = \{HHH, HHT, THH, HTH\}$

$P(A) = \frac{4}{8} = \frac{1}{2}$

$= 0.5$

(4) When 4 coins are tossed, find the Probability of getting equal no.of heads & tails.

sol

$$S = \begin{Bmatrix} HHHH & HTHH & THHH & TTHH \\ HHHT & HTHT & THHT & TTHT \\ HHTH & HTTH & THTH & TTTH \\ HHTT & HTTT & THTT & TTTT \end{Bmatrix}$$

Event A = Getting equal no.of heads & tails

$P(A) = \frac{6}{16} = 3/8$

$P(A) = 0.37$

(5) When a Single die thrown. what is the Probability of getting Even number

sol

$S = \{1, 2, 3, 4, 5, 6\}$

Event A = Getting a Even number

$A = \{2, 4, 6\}$

$P(A) = 3/6$

$P(A) = 0.5$

(6) When two dice thrown, what is the Probability of getting Sum of 2 dice more than 5.

Sol

$$S = \left\{ \begin{array}{cccccc} 1,1 & 2,1 & 3,1 & 4,1 & 5,1 & 6,1 \\ 1,2 & 2,2 & 3,2 & 4,2 & 5,2 & 6,2 \\ 1,3 & 2,3 & 3,3 & 4,3 & 5,3 & 6,3 \\ 1,4 & 2,4 & 3,4 & 4,4 & 5,4 & 6,4 \\ 1,5 & 2,5 & 3,5 & 4,5 & 5,5 & 6,5 \\ 1,6 & 2,6 & 3,6 & 4,6 & 5,6 & 6,6 \end{array} \right\}$$

Event A = Getting Sum is more than 5

$$A = \left\{ \begin{array}{ccccccc} 1,5 & 2,6 & 3,6 & 4,5 & 5,3 & 6,1 & 6,5 \\ 1,6 & 3,3 & 4,2 & 4,6 & 5,4 & 6,2 & 6,6 \\ 2,4 & 3,4 & 4,3 & 5,1 & 5,5 & 6,3 & \\ 2,5 & 3,5 & 4,4 & 5,2 & 5,6 & 6,4 & \end{array} \right\}$$

$P(A) = \frac{26}{36} = 13/18$

$= 0.72$

(7) When two dice of thrown, Find the Probability of getting Sum of two dice, the number greater than 10.

Sol

$$S = \left\{ \begin{array}{cccccc} 1,1 & 2,1 & 3,1 & 4,1 & 5,1 & 6,1 \\ 1,2 & 2,2 & 3,2 & 4,2 & 5,2 & 6,2 \\ 1,3 & 2,3 & 3,3 & 4,3 & 5,3 & 6,3 \\ 1,4 & 2,4 & 3,4 & 4,4 & 5,4 & 6,4 \\ 1,5 & 2,5 & 3,5 & 4,5 & 5,5 & 6,5 \\ 1,6 & 2,6 & 3,6 & 4,6 & 5,6 & 6,6 \end{array} \right\}$$

$A = \{5,6 \quad 6,5 \quad 6,6\}$

$P(A) = 3/36$

$= 1/12$

(8) In a Single thrown of 3 dice, what is the Probability of getting Same number on 3 dice.

Sol S = Total outcomes is 216

Event A = Getting a Same number of 3 dice

$A = \{111, 222, 333, 444, 555, 666\}$

$A = 6/216$

$A = 0.0277$

State Baye's theorem:-

It states that if a sample space S has N mutually exclusive events B_n, $n = 1, 2, \ldots, N$, such that $B_m \cap B_n = \{\phi\}$ for $m \neq n = 1, 2, \ldots, N$, and any event A is defined on this sample space, then the conditional probability of B_m and A can be written as

$$P(B_n|A) = \frac{P(A|B_n)P(B_n)}{P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \ldots + P(A|B_N)P(B_N)}$$

(or)

$$P(A \cap B) > P(A)P(B)$$

Thus, $$P(B|A) = \frac{P(A \cap B)}{P(A)} > \frac{P(A)P(B)}{P(A)} = P(B)$$

(or) $P(B|A) > P(B)$

Total probability theorem:-

Consider a sample space S that has N mutually exclusive events B_n, $n = 1, 2, \ldots, N$, such that $B_m \cap B_n = \{\phi\}$ for $m \neq n = 1, 2, \ldots, N$. The probability of any event A, defined on this

Sample space can be expressed in terms of the conditional probabilities of event B_n. This probability is known as the total probability of event A

mathematically,

$$P(A) = \sum_{n=1}^{N} P(A|B_n)P(B_n)$$

Random variable:-

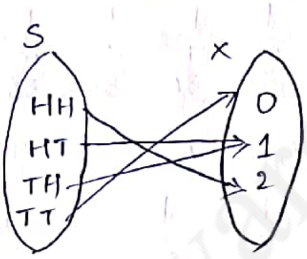
Random variable is a function that assigns real values to all the outcomes (s) in the sample space (S) of a random experiment.

Eg-1:- consider an experiment of tossing a coin twice

$S = \{HH, HT, TH, TT\}$

Let 'x' be a random variable such that it takes total no. of heads of the experiment

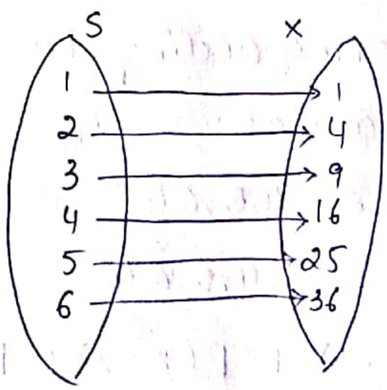
$x = \{0,1,2\}$



Eg:-2:- consider an experiment of throwing a die. Let the random variable 'x' takes the values of squares of the outcomes of sample space

$S = \{1,2,3,4,5,6\}$

$x = \{1,4,9,16,25,36\}$



Eg:-3 Two dice are thrown the square of the sum of the points appearing on the two dice is a random variable 'x'. determine the values taken by 'x' and the corresponding probabilities

$S = \{(1,1)\ (1,2)\ (1,3)\ (1,4)\ (1,5)\ (1,6)$
$(2,1)\ (2,2)\ (2,3)\ (2,4)\ (2,5)\ (2,6)$
$(3,1)\ (3,2)\ (3,3)\ (3,4)\ (3,5)\ (3,6)$
$(4,1)\ (4,2)\ (4,3)\ (4,4)\ (4,5)\ (4,6)$
$(5,1)\ (5,2)\ (5,3)\ (5,4)\ (5,5)\ (5,6)$
$(6,1)\ (6,2)\ (6,3)\ (6,4)\ (6,5)\ (6,6)\}$

$X = \{2, 5, 10, 17, 26, 37, 8, 13, 20, 29, 40, 18, 25, 30, 45, 32, 34, 52, 50, 61, 72\}.$

$X = \{$

2	5	8	10	13	17	18	20	25	26	29	32	34
$\frac{1}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$

37	40	41	45	50	52	61	72 $\}$
$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

Conditions for a function to be a random variable:-

1. It can be any function but not multi value
2. The set $\{X \le x\}$ corresponds to all the events $X \le x$

Probability of $X \le x$ is equal to the probabilites of all the events corresponding to $X \le x$.

Eg:- Throwing a die

$S = \{1,2,3,4,5,6\}$

$X = \{1,2,3,4,5,6\}$

$$P(X \le 4) = P(X=1) + P(X=2) + P(X=3) + P(X=4)$$

3. Probability of $X = \infty$

$$P(X=\infty)=0,\quad P(X=-\infty)=0$$

Types of random variable:-

① discrete random variable:-

A random variable 'x' is said to be a discrete random variable. If it takes finite number of discrete values in a given interval

Eg:- No. of defectives in a given sample of electric Bulb.

No. of printing mistakes in a page

② Continuous random variable:-

A random variable 'x' is said to be continuous random variable. If it takes infinite no. of values in a given interval

Eg:- Height of a person entering the room

Time

Temperature

③ mixed random variable:-

If a random variable 'x' takes both discrete and continuous values then it is called mixed random variable.

Eg:- No. of telephone calls received in a particular time interval. Here No. of calls is discrete and time of receiving calls is continuous

Probability distribution function [PDF]

(or)

cumulative distribution function [CDF]

If 'x' is a random variable then the probability of $(x \le x)$ is called as probability distribution function (PDF). It is denoted by the letter $F_X(x)$.

It denotes the sum of probability of all the events corresponding to $(x \le x)$ for any value of 'x'

$$\boxed{F_X(x) = P(X \le x)}$$

Properties of Probability distribution function (PDF):-

1. $F_X(-\infty)=0$

$F_X(x) = P(X \leq x)$

$F_X(-\infty) = P(X \leq -\infty)$

In a random variable all the assigned real numbers are vary from $-\infty$ to $+\infty$ only that is $(-\infty < X < +\infty)$. There is no real numbers less than $-\infty$

$\therefore P(X \leq -\infty) = 0$

2. $F_X(\infty)=1$

$F_X(x) = P(X \leq x)$

$F_X(\infty) = P(X \leq \infty)$

$= P(X=-\infty) + \text{------} + P(X=0) + \text{------} + P(X=\infty)$

$= \sum_{x=-\infty}^{\infty} P(X=x)$

$= 1$

3. $0 \leq F_X(x) \leq 1$

Since the probability distribution function PDF $(F_X(x))$ takes ~~to~~ all the probabilities, we know that the range of probability is always from 0 to 1.

Therefore the range of PDF is also from 0 to 1.

4. If $x_1 < x_2$ then $F_X(x_1) < F_X(x_2)$

Since the values of probability are always Non-Negative, therefore the PDF is always monotoni-cally increasing function.

Therefore if 'x' increases then $F_X(x)$ is also increases

5. $F_X(x^+) = F_X(x)$

where as x^+ is infinitesimely small increment of 'x'

Eg:- Rolling a dice

$S = \{1,2,3,4,5,6\}$

$X = \{1,2,3,4,5,6\}$

$P(x=x) = \{1/6, 1/6, 1/6, 1/6, 1/6, 1/6\}$ ⑧

$F_X(3) = P(x \le 3)$

$= P(x=1) + P(x=2) + P(x=3)$

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \Rightarrow 3/6$

~~F(x) (3.1)~~ $F_X(3.1) = P(x \le 3.1)$

$= P(x=1) + P(x=2) + P(x=3)$

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \Rightarrow 3/6$

6. $P(x_1 < x \le x_2) = F_X(x_2) - F_X(x_1)$

$P(x \le x_2) = P(x \le x_1) + P(x_1 < x \le x_2)$

$P(x_1 < x \le x_2) = P(x \le x_2) - P(x \le x_1)$

$= F_X(x_2) - F_X(x_1)$

Eg:- consider an experiment of throwing a die. Let 'x' takes the values of the outcomes of an experiment

$S = \{1,2,3,4,5,6\}$

$x = \{1,2,3,4,5,6\}$

$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow$

$\frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6} \quad \frac{1}{6}$

PDF $F_X(x) = P(x \le x)$

$F_X(1) = P(x \le 1)$

$= P(X=1)$

$= \frac{1}{6}$

$F_X(2) = P(x \le 2)$

$= P(x=1) + P(x=2)$

$= \frac{1}{6} + \frac{1}{6} \Rightarrow \frac{2}{6} \Rightarrow \frac{1}{3}$

$F_X(3) = P(x \le 3)$

$= P(x=1) + P(x=2) + P(x=3)$

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \Rightarrow \frac{3}{6} \Rightarrow \frac{1}{2}$

$F_X(4) = P(X \le 4)$

$= P(X=1) + P(X=2) + P(X=3) + P(X=4)$

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$

$= \cancel{\frac{4}{6}} \frac{2}{3} \Rightarrow \frac{2}{3}$

$F_X(5) = P(X \le 5)$

$= P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$

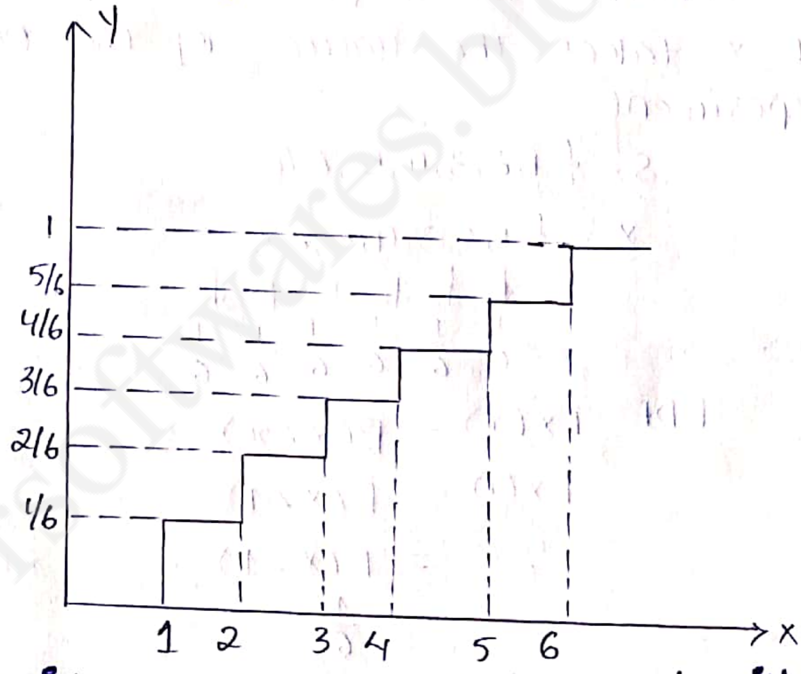
$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \cancel{+ \frac{1}{6}}$

$\Rightarrow \frac{5}{6}$

$F_X(6) = P(X \le 6) \Rightarrow P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5) + P(X=6)$

$= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$

$= \frac{6}{6} \Rightarrow 1$



Eg:- Consider a Random Variable 'X' with the following data

$X = x$	0	1	2	3	4	5	6	7	8
$P(X=x)$	a	3a	5a	7a	9a	11a	13a	15a	17a

(i) find a (ii) $P(X \le 3)$ (iii) $(P X > 3)$ (iv) $P(a < X \le 5)$

(i) $a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$

$15a + 10a + 30a + 26a = 1$

$\Rightarrow 70a + 11a = 1$

$81a = 1 \quad \Rightarrow a = 1/81$

$\cancel{a = 81} \quad \Rightarrow 0.012$

(ii) $P(X \leq 3)$

$= P(X=1) + P(X=2) + P(X=3) + P(X=0)$

$= 3a + 7a + 5a + a$

$= 10a + 6a \Rightarrow 16a$

$\Rightarrow 16(0.012) = 0.192$

(iii) $P(X > 3)$

$= P(X=4) + P(X=5) + P(X=6) + P(X=7) + P(X=8)$

$= 9a + 11a + 13a + 15a + 17a$

$= 30a + 26a + 9a$

$= 59a + 6a \Rightarrow 65a \Rightarrow 65(0.012) \Rightarrow 0.74.$

(iv) $P(0 < x \leq 5)$

$= P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)$

$= 3a + 5a + 7a + 9a + 11a$

$= 20a + 10a + 5a$

$\Rightarrow 35a \Rightarrow 35(0.012) \Rightarrow 0.42$

Probability density function (pdf)

(or)

Probability mass function

Probability density function is the derivate of probability distribution function (PDF)

It is denoted by

$$\boxed{f_X(x) = \frac{d}{dx} F_X(x)}$$

Probability density function of a discrete random variable often known as probability mass function

Properties of density function:-

1. $f_X(x) \geq 0$

since pdf is the derivate of the distribution function and distribution function monotonically increasing function

Therefore, density function is always Non-Negative

2. $\int_{-\infty}^{\infty} f_X(x)\, dx = 1$

If we integrate $f_X(x)$, we will get $F_X(x)$

$$\int_{-\infty}^{\infty} f_X(x)\, dx = [F_X(x)]_{-\infty}^{\infty}$$

$$= F_X(\infty) - F_X(-\infty)$$

$$= 1 - 0$$

$$= 1$$

The above two properties can be used to check whether the given function is a valid pdf (or) not

3. $\int_{-\infty}^{x} f_X(x)\, dx = F_X(x)$

If we integrate $f_X(x)$ from $-\infty$ to x

$$\int_{-\infty}^{x} f_X(x)\, dx = [F_X(x)]_{-\infty}^{x}$$

$$= F_X(x) - F_X(-\infty)$$

$$= F_X(x) - 0$$

$$\boxed{\int_{-\infty}^{x} f_X(x)\, dx = F_X(x)}$$

4. $\int_{x_1}^{x_2} f_X(x)\, dx$

If we integrate $f_X(x)$ from x_1 to x_2

$$\int_{x_1}^{x_2} f_X(x)\, dx = F_X(x_2) - F_X(x_1)$$

from probability distribution function property

$$P(x_1 < X \le x_2) = F_X(x_2) - F_X(x_1)$$

$$\boxed{\int_{x_1}^{x_2} f_X(x)\, dx = P(x_1 < X \le x_2)}$$

Problem

1) A random variable x as pdf $f_X(x) = \begin{cases} c(1-x^4) & -1 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

① Find c ② Find $P(|x| < 1/2)$ ③ PDF

Sol:-

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1$$

$$\int_{-\infty}^{\infty} c(1-x^4)\,dx = 1$$

$$c\left[x - \frac{x^5}{5}\right]_{-1}^{1} = 1$$

$$c\left[\left((1) - \frac{(1)^5}{5}\right) - \left((-1) - \frac{(-1)^5}{5}\right)\right] = 1$$

$$c[1 - 1/5 - ((-1) - ((-1)/5))] = 1$$

$$c[4/5 - (-1 + 1/5)] = 1$$

$$c[4/5 + 4/5] = 1 \Rightarrow c[8/5] = 1$$

$$\therefore c = 5/8$$

$$\int_{-1/2}^{1/2} c(1-x^4)\,dx$$

$$f(x) = \int_{-1/2}^{1/2} \frac{5}{8}(1-x^4)\,dx$$

$$= 5/8\left[x - \frac{x^5}{5}\right]_{-1/2}^{1/2}$$

$$= 5/8\left[\left(1/2 - \frac{(1/2)^5}{5}\right) - \left(-1/2 - \frac{(-1/2)^5}{5}\right)\right]$$

$$= 5/8\left[(1/2 - 1/60) - (-1/2 - (-1/160))\right]$$

$$= 5/8[1/2 - 1/60 + 1/2 - 1/60]$$

$$= 5/8 \times 2(1/2 - 1/60)$$

$$= 0.61$$

PDF

$$F_X(x) = \int f_X(x)\,dx$$

$$= 5/8\left[x - \frac{x^5}{5}\right]$$

2) Consider an Experiment of tossing a Coin trice. if random Variable 'x' represents no.of tails plot. probability density Function.

Sol

$S = \{HHH, HHT, HTH, THH, TTH, THT, TTH, TTT\}$

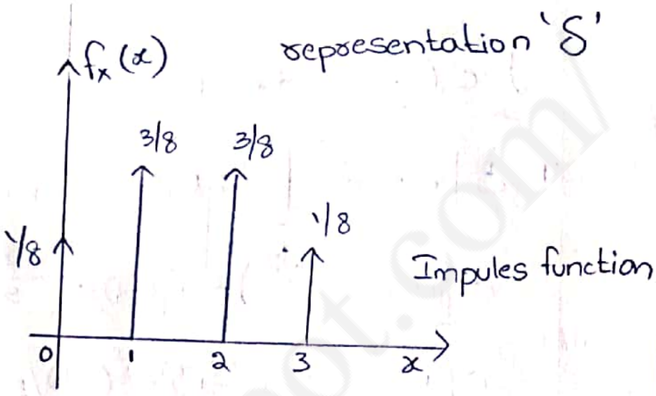
x = No. of Tails.

$x = \{0, 1, 2, 3\}$

$P(X=x) \Rightarrow$

$P(x=0) = 1/8$, $P(x=1) = 3/8$,

$P(x=2) = 3/8$, $P(x=3) = 1/8$



Standard distribution and density function :-

① Binomial distribution function
② Poission distribution function
} Discrete random Variable

③ Gaussion distribution function
④ Uniform distribution function
⑤ Exponential distribution function
⑥ Rayleigh distribution function
} Continuous random Variable.

1. Binomial distribution Function :-

It is useful in the experiments where there are two Outcomes only, for example yes (or) No, 1 or 0, Heads (or) Tails.

Suppose if we perform a Series of independent trails Such that 'P' represents the probability of Success and 'q' represent is that of failure. The probability of getting 'x' no. of success for 'n' independent trails is given by

$$\boxed{P(X=x) = n_{C_x} (P)^x (q)^{n-x}}$$

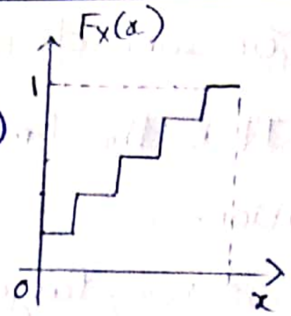
P = Prob. of Success

q = Prob. of failure

$$\boxed{q = 1-P}$$

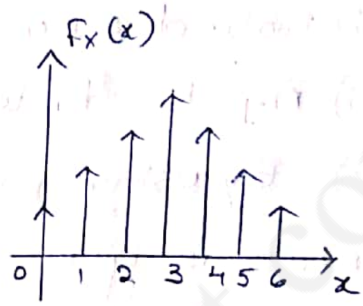
$F_X(x) = P(X \le x)$

$= P(X=0) + P(X=1) + \cdots\cdots + P(X=x)$



$$\boxed{F_X(x) = \sum_{k=0}^{n} {}^{n}C_k \, (p)^k \, (q)^{n-k} \, u(x-k)}$$

Bionomial distribution function



$$\boxed{f_X(x) = \sum_{k=0}^{n} {}^{n}C_k \, (p)^k \, (q)^{n-k} \, \delta(x-k)}$$

Bionomial density function.

Ex: Consider an Experiment of tossing a coin 6 times find the Probability of getting 4 heads.

Sol:

$n=6$ = no. of times the trail Conducted

$x=4$ = no. of times heads required

$p=1/2$ = Probability of getting head

$q=1/2$ = Probability of getting tail

$P(X=x) = {}^{n}C_x \, (p)^x \, (q)^{n-x}$

$= {}^{6}C_4 \, (1/2)^4 \, (1/2)^{6-4}$

$= {}^{6}C_4 \, (1/16) \, (1/2)^2$

$= \frac{6 \cdot 5}{2 \cdot 1} \cdot \frac{1}{16} \cdot \frac{1}{4}$

$= 15 \cdot \frac{1}{16} \cdot \frac{1}{4}$

$= 15/64$

$= 0.234$

$[{}^{n}C_r = {}^{n}C_{n-r}]$

${}^{n}C_x = \frac{n!}{(n-x)!\,x!}$

${}^{n}C_x = \frac{6!}{(6-4)!\,4!}$

Applications:

This Function can be applied in many problem delection in Radar.

2. Poisson distribution function :-

It is the limiting Case of binomial distribution under the Condition.

i) n is very large $n \to \infty$

ii) Prob. of success very small $p \to 0$

iii) $n.p = b$ finite value

iv) By using poission distribution Prob. of getting 'x' no. of success is given by prob. of $(x = x)$

$$\boxed{P(x=x) = \frac{e^{-b} \cdot b^{x}}{x!}}$$

Poission Prob. distribution function :

$$\boxed{F_x(x) = \sum_{k=0}^{\infty} \frac{e^{-b} b^{k}}{k!} u(x-k)}$$

Poisson Prob. density function :

$$\boxed{f_x(x) = \sum_{k=0}^{\infty} \frac{e^{-b} b^{k}}{k!} \delta(x-k)}$$

Applications:

→ It is mostly applied in Counting type problem

(1) No. of telephone calls made in a given time

(2) No. of defective elements in a given sample

Problem: If 20% of the bolts manufactured by a company are defective find the probability that two bolts out of '10' are the defective by using i) Binomial distribution

ii) Poisson distribution

Sol

$n = 10$

$P = \frac{20}{100} = 0.2$

$q = 0.8$

$P(x=2) = nC_x \cdot p^{x} \cdot q^{n-x}$

$$= {}^{10}C_2 \, (0.2)^2 (0.8)^{10-2}$$

$$= \frac{\overset{5}{\cancel{10}} \times 9}{1 \times \cancel{2}} (0.2)(0.8)^8$$

$$= 0.3019$$

b) Poisson

$$P(x=x) = \frac{e^{-b} b^x}{x!}$$

$$b = n \cdot p = (10)(0.2)$$

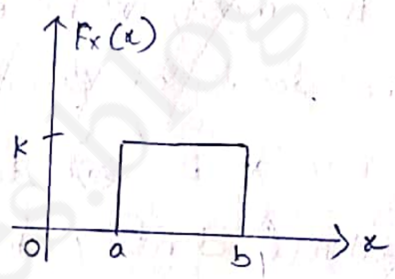
$$= 2$$

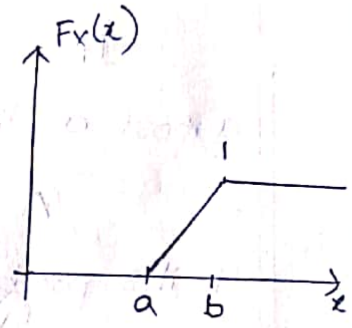
$$P(x=2) = \frac{e^{-2}(2)^2}{2!} = 0.2706$$

Uniform distribution function :-

A RV, x is said to be uniformly distributed over an interval (a,b) if it's probability density function is constant 'k' over the interval (a,b)

$$\boxed{\begin{aligned} f_x(x) &= k \quad a \le x \le b \\ &= 0 \quad \text{otherwise} \end{aligned}}$$





$$\int_{-\infty}^{\infty} f_x(x)\,dx = 1$$

$$\int_a^b k\,dx = 1$$

$$k[x]_a^b = 1$$

$$k[b-a] = 1$$

$$k = 1/b-a$$

$$F_x(x) = \int_{-\infty}^{x} f_x(x)\,dx$$

$$= \int_{-\infty}^{x} \frac{1}{b-a}\,dx$$

$$= \int_{-\infty}^{x} (b-a)^{-1}\,dx$$

Uniform density function

$$\boxed{\begin{aligned} f_x(x) &= \frac{1}{b-a} \quad a \le x \le b \\ &= 0 \quad \text{Otherwise} \end{aligned}}$$

Uniform distribution function

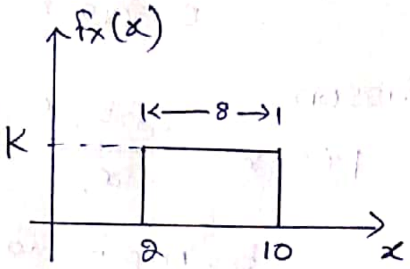
$$\boxed{\begin{aligned} F_x(x) &= 0 \quad \text{if } x < a \\ &= \frac{x-a}{b-a} \quad \text{if } a \le x < b \\ &= 1 \quad \text{if } x > b \end{aligned}}$$

Application:

Digital Communication (Sampling & Quantization process)

Find the value of 'k' if R.V 'x' is uniformly distributed in the interval (2,10) having constant value of 'k'.

$f_x(x) = 1$



$\int_{-\infty}^{\infty} f_x(x)\,dx = 1$

$\int_{2}^{10} k\,dx = 1$

$k[x]_2^{10} = 1$

$k[10-2] = 1$

$k[8] = 1 \Rightarrow 1/8$

Gaussian distribution function [Normal distribution function]:-

It is the most useful continuous distribution function in almost all the areas of science and Engineering.

A RV 'x' is said to be having Gaussian distribution if its probability density function is given by

$$\boxed{f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}}\, e^{\frac{-(x-a_x)^2}{2\sigma_x^2}}}$$

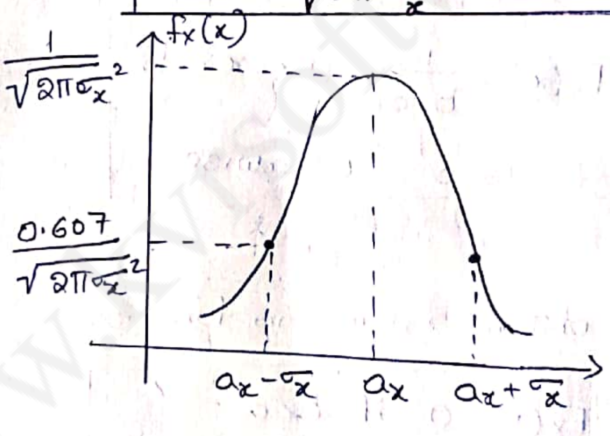
where, $-\infty < x < \infty$

a_x = Mean

σ_x^2 = Variance

σ_x = Standard deviation

$\sigma_x > 0$



For max. Value

Substitute $x = a_x$

$$f_x(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}}\, e^{\frac{-(a_x-a_x)^2}{2\sigma_x^2}}$$

$$= \frac{1}{\sqrt{2\pi\sigma_x^2}}\, e^{\frac{-0}{2\sigma^2}} \Rightarrow \frac{1}{\sqrt{2\pi\sigma_x^2}} \quad (1)$$

For Probability to get Symmetric Value

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{\frac{-(a_x-\sigma_x-a_x)^2}{2\sigma_x^2}}$$

$$= \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{\frac{-\sigma_x^2}{2\sigma_x^2}}$$

$$= \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-1/2}$$

$$= \frac{1}{\sqrt{2\pi\sigma_x^2}} (0.607) \Rightarrow \frac{0.607}{\sqrt{2\pi\sigma_x^2}}$$

Characteristics of Gaussian distribution function :-

1) It is a bell shaped curve

2) It is having maximum Value of $\frac{1}{\sqrt{2\pi\sigma_x^2}}$ at $x=a_x$

3) It is Symmetrical about the mean value a_x

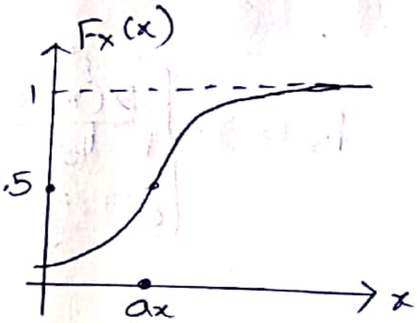
4) Area under the density curve is '1'

5) It is having a magnitude of 0.607 times the maximum Value at $x=a_x-\sigma_x$ and $x=a_x+\sigma_x$

Probability distribution function :-

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

$$= \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma_x^2}} \cdot e^{\frac{-(x-a_x)^2}{2\sigma_x^2}}\,dx$$



$$\boxed{F_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \int_{-\infty}^{x} \frac{-(x-a_x)^2}{2\sigma_x^2}\,dx}$$

Gaussian distribution function

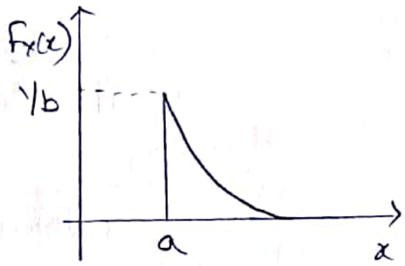
Application:-

1) In Electronic Communication System the noise is characteri--sed by using a Gaussian density function.

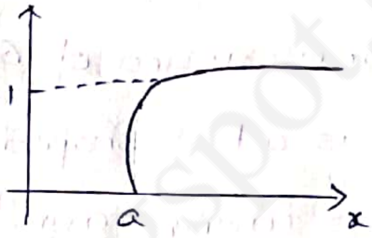
Exponential distribution function :-

A R.V 'x' is said to be Exponentially distributed if it is having probability density function as.

$$f_X(x) = \begin{cases} \frac{1}{b} e^{-\frac{(x-a)}{b}} & \text{for } x \geq a \\ 0 & \text{for } x < a \end{cases}$$



$$F_X(x) = \begin{cases} 1 - e^{-\frac{(x-a)}{b}} & x \geq a \\ 0 & x < a \end{cases}$$

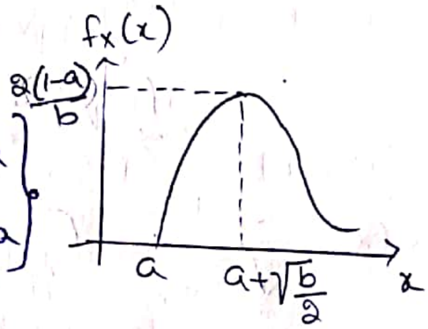


Applications:-

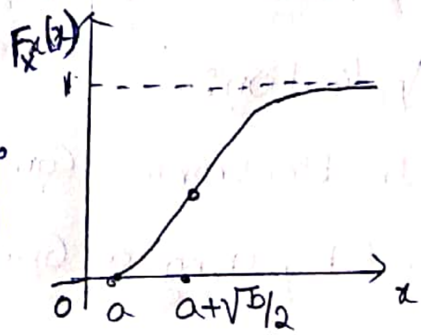
* In hydrology, the Exponential function is used to analyze Extreme value of such variable as monthly & annual maximum values of daily rainfall
* Exponential distribution function describes the time for a Continuous process to changes state.

Rayleigh Distribution function :-

$$f_X(x) = \begin{cases} \frac{2(x-a)}{b} e^{-\frac{(x-a)^2}{b}} & x \geq a \\ 0 & x < a \end{cases}$$



$$F_X(x) = \begin{cases} 1 - e^{-\frac{(x-a)^2}{b}} & x \geq a \\ 0 & x < a \end{cases}$$



Applications:-

1. In communication theory
2. To model multiple paths of dense scattered signals reacting a receivers

Conditional Probability:-

Consider an experiment with A and B outcomes. The probability of the outcome 'B' given that 'A' is known is called conditional probability

$$P(A) = P(B|A) = \frac{P(A \cdot B)}{P(A)} \quad (\text{or}) \quad \frac{P(A \cap B)}{P(A)}$$

$$P(B) = P(A/B) = \frac{P(A \cdot B)}{P(B)} \quad (\text{or}) \quad \frac{P(A \cap B)}{P(B)}$$

Conditional distribution function:-

$$F_X(x/B) = P((X \leq x)/B)$$

Properties:-

1. $F_X(-\infty/B) = 0$
2. $F_X(\infty|B) = 1$
3. $0 \leq F_X(x|B) \leq 1$
4. If $x_1 < x_2$ then $F_X(x_1/B) < F_X(x_2/B)$
5. $P[(x_1 < X \leq x_2)/B] = F_X(x_2/B) - F_X(x_1/B)$
6. $F_X(x^+/B) = F_X(x/B)$

Conditional density function:-

$$f_X(x/B) = \frac{d}{dx} F_X(x/B)$$

Properties:-

1. $f_X(x) \geq 0$
2. $\int_{-\infty}^{\infty} f_X(x/B) = 1$
3. $F_X(x/B) = \int_{-\infty}^{\infty} f_X(x|B)\,dx$
4. $P(x_1 < X < x_2)/B = \int_{x_1}^{x_2} f_X(x|B)\,dx$

Summary :-

1. Introduction → deterministic signal
→ Random signal

2. Review of Probability
→ sample space
→ event → certain event
→ Impossible event
→ mutually exclusive event
→ Independent event

3. Probability → relative frequency
→ Axiomatic approach

4. Random variable → discrete
→ continuous
→ mixed
→ conditions to be a random variable

5 Probability distribution function (PDF)
(or) Cumulative distribution function (CDF)
[$F_X(x) = P(X \leq x)$] → properties

6. Probability density function (pdf)
(or) Probability mass function (pmf)
[$f_X(x) = \frac{d}{dx} F_X(x)$]

7. standard distribution function
→ Binomial → Gaussian
→ Poisson → uniform
→ exponential → Rayleigh

8. Conditional distribution function
$F_X(x|B) = P(X \leq x|B)$ → Properties

Conditional density function
$f_X(x|B) = \frac{d}{dx}\left(F_X(x|B)\right)$ → properties

18/07/17 2. operation on one Random variable - Expectation

Expectation:-

Expectation is the name given to the random process of averaging when a random variable is involved.

for a random variable 'x' the expected value can be represented as '$E(x)$'.

i.e.. it may also called as the expectation of x, The expected value of x, the mean value of x. (or) statistical average value of x

in other terms the expected value may also written as

$$E(x) = \overline{x}$$

Expected value of a random variable:-

If x be a discrete random variable then it has discrete values of x_i that occur it probability $P(x_i)$ and can be represented as

$$E(x) = \overline{x}$$

$$= \sum_{i=1}^{N} x_i P(x_i)$$

where X is a discrete random variable

$1 < X < N$

if we consider x_i with the fractional dollar values while $P(x_i)$ is the ratio of no. of people for the given 100 dollar value to the total no. of people.

$$E(x) = \sum_{i=1}^{100} x_i P(x_i)$$

In general the expected value of a random variable x can be defined as

$$E(x) = \bar{x}$$

$$\boxed{E(x) = \int_{-\infty}^{\infty} x f_X(x) dx} \quad \text{--- (1)}$$

where $f_X(x)$ is called density function

If x happens to ~~the~~ discrete with N possible values of 'x_i' having probabilities $P(x_i)$ then

$$\boxed{f_X(x) = \sum_{i=1}^{N} P(x_i) \delta(x - x_i)} \quad \text{--- (2)}$$

i.e., from the above two equations the expected value random variable can be given as

$$E(x) = \sum_{i=1}^{N} x_i P(x_i)$$

i.e., the expected value of random variable x

expected value of a function of a random variable:

we know that the expected value of a discrete random variable can be given as

$$\bar{x} = E(x) = \sum_{i=1}^{N} x_i P(x_i)$$

in continuous form it may also represented as

$$E(x) = \bar{x} = \int_{-\infty}^{\infty} x f_x(x)\, dx$$

based on this two values the expected value of the real function g(x) can be given as

$$E[g(x)] = \sum_{i=1}^{N} g(x_i) \cdot P[g(x_i)] \rightarrow \text{if } g(x) \text{ is discrete}$$

$$E[g(x)] = \int_{-\infty}^{\infty} g(x) f_x(x)]\, dx \rightarrow$$

if g(x) is continuous

problem :-

1) 90 people are randomly selected and the fractional doller value of coin in their pockets is counted, if they count base about a doller, the doller value is discorded and only the portion from 0 to 99 is accepted.

It is found that, 8, 12, 28, 22, 15, 5 people had 18, 45, 64, 72, 77 and 95 in their pockets respectively find the average value.

A)

No. of people	8	12	28	22	15	5
No. of coins	18	45	64	72	77	95
	0.18	0.45	0.64	0.72	0.77	0.95

$$\bar{X} = E[X] = \sum_{i=1}^{N} x_i P(x_i)$$

$$\Rightarrow \sum_{i=1}^{6} x_i P(x_i)$$

$$= 0.18\left(\frac{8}{90}\right) + 0.45\left(\frac{12}{90}\right) + 0.64\left(\frac{28}{90}\right) + 0.72\left(\frac{22}{90}\right) + 0.77\left(\frac{15}{90}\right) + 0.95\left(\frac{5}{90}\right)$$

$$= 0.18(0.08) + 0.45(0.13) + 0.64(0.31) + 0.72(0.24) + 0.77(0.16) + 0.95(0.05)$$

$0.014 +$

$= 0.016 + 0.06 +$

$=$

conditional expected

for a random variable 'x' the conditional density $f_x(x/B)$ where B is any event in the sample space then the conditional expected value can be given as $E[x] = \bar{x} = \int_{-\infty}^{\infty} x f_x(x) dx$

like this the conditional Expected value can be given as $\boxed{E[x/B] = \int_{-\infty}^{\infty} x f_x(x/B) dx}$ ——— ①

where B is an event

$$B = \{X \leq b\} \text{ and } -\infty \leq b < \infty$$

By this function $f_x(x \leq b)$ can be given as

$$f_x(x/x \leq b) = \frac{f_x(x)}{\int_{-\infty}^{b} f_x(x) dx}, \text{ if } x \leq b \quad ——— ②$$

$= 0$ else

Eq ② in Eq ①

$$E(X/B) = \int_{-\infty}^{\infty} \frac{f_X(x)}{\int_{-\infty}^{b} f_X(x)\,dx}$$

where $F_X(x)$ = Probability $P\{X \le x\}$

and $F_X(x/B) = P\{x / x \le b\}$

Moments :-

Basically the moments can be classified into two categiries.

1. moment about the origin
2. central moments (moments about the mean)

Moment about the origin :-

The function $E(X) = \int_{-\infty}^{\infty} x f_X(x)\,dx$ is called moment about the origin of a random variable 'x' if $g(x) = x^n$

where $n = 0, 1, 2, 3 \ldots$

And it is denoted by m_n

$$\therefore \boxed{m_n = E(X^n) = \int_{-\infty}^{\infty} x^n f_X(x)\,dx}$$

for $n = 0$

$$m_0 = E[x^0] = \int_{-\infty}^{\infty} x^0 f_X(x)\,dx$$

$$m_0 = \int_{-\infty}^{\infty} f_X(x)\,dx$$

that is area of density function

if $n=1$

$$m_1 = E[x^1] = \int_{-\infty}^{\infty} x^1 f_X(x)\,dx$$

$$= \int_{-\infty}^{\infty} x f_X(x)\,dx$$

that is expected value of a random variable x

central moments :- $[\mu_n]$

Moments about the mean value of x is called central moments and are denoted by μ_n. These are defined as the expected value of a function $g(x) = (x-\bar{x})^n$

$$\therefore \boxed{\mu_n = E(x-\bar{x})^n = \int_{-\infty}^{\infty} (x-\bar{x})^n f_X(x)\,dx}$$

where μ_n is called n^{th} order central moment, where $n = 0, 1, 2, \ldots$

varience (σ_x^2):-

The second central moment μ_2 is named as varience and is denoted with a special notation "σ_x^2" then the varience can be given as

$$\mu_2 = \sigma_x^2$$

$$= E[x-\bar{x}]^2$$

$$\boxed{\mu_2 = E[x-\bar{x}]^2 = \int_{-\infty}^{\infty} (x-\bar{x})^2 f_x(x)\,dx}$$

Note:-

standard deviation:-

The +ve square root of varience is called standard deviation and is denoted by "σ_x"

$$\sigma_x = \sqrt{\sigma_x^2}$$

Skew:- $E[x-\bar{x}]^2 = E[x^2+\bar{x}^2-2x\bar{x}]$

$$= E[x^2]+E[\bar{x}^2]-2E[x\bar{x}]$$

$$\Rightarrow m_2+\bar{x}^2-2\bar{x}^2$$

$$\Rightarrow m_2-\bar{x}^2$$

$$\Rightarrow m_2-(\bar{x})^2$$

$$\Rightarrow m_2-(m_1)^2$$

skew (σx^3):

The third central moment is called skew of a random variable 'x'. and can be denoted as. σx^3 or μ_3

$$\sigma x^3 = \mu_3 = E[x - \bar{x}]^3$$

$$\therefore \sigma x^3 = \int_{-\infty}$$

$$\boxed{\sigma x^3 = E[x-\bar{x}]^3 = \int_{-\infty}^{\infty} (x-\bar{x})^3 f_x(x)\, dx}$$

skewness:

The normalize third central moment is called skewness. i.e. $\frac{\mu_3}{\sigma x^3}$ is called skewness of the density function.

20/3/17 function that gives moments:

two functions can be define that allow moments to be calculated for a random variable 'x'. These are

1, these are charcteristic function

2, moment generating function

i) characteristic function:-

The characteristic function of a random variable 'x' is defined by $\boxed{\phi_x(\omega) = E[e^{j\omega x}]}$ —— ①

where $j = \sqrt{-1}$ ~~why~~ and $-\infty < \omega < \infty$.

The characteristic function in terms of density function can be represented as

$$\boxed{\phi_x(\omega) = E[e^{j\omega x}] = \int_{-\infty}^{\infty} f_x(x) e^{j\omega x}\, dx} \quad \text{—— ②}$$

i.e., $\phi_x(\omega)$ is seen to be the fourier transform of $f_x(x)$ and it can be given as

$$\boxed{f_x(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_x(\omega) \cdot e^{-j\omega x}\, d\omega} \quad \text{—— ③}$$

from Eq ② the n^{th} order moment of characteristic function can be obtained as differentiation of $\phi_x(\omega)$ w.r.to ω of n times and $\omega = 0$

$$m_n = \left.\frac{d^n \phi_x(\omega)}{d\omega^n}\right|_{\omega=0}$$

Note:- The maximum magnitude of characteristic function is unity and occurs at $\omega = 0$

i.e., $|\phi_x(\omega)| \le |\phi_x(0)| = 1$ $\left[\because \int_{-\infty}^{\infty} f_x(x)dx = 1\right]$

moment generating function:

The moment generating function is denoted by $M_X(v)$

$\therefore M_X(v) = E[e^{vX}]$

i.e., $\boxed{M_X(v) = \int_{-\infty}^{\infty} e^{vx} f_X(x)\,dx}$

where 'X' is a random variable

v is a real number i.e. $-\infty < v < \infty$

The main advantage is that it can give the moments related to $M_X(v)$ and can be expressed as

$$M_n = \frac{d^n}{dv^n} M_X(v)\Big|_{v=0}$$

disadvantages:-

It can't exist for all values of v and it exist all values of v in neighbour hood of $v=0$ that is called moment generating function.

17 Chernoff inequality & bound:-

This is an important application of moment generating function.

Ex: If 'x' may be random variable, non -ve and for all $v>0$

$$\therefore \boxed{e^{v(x-a)} \geq u(x-a)} \quad \text{--- (1)}$$

we know that

$$\boxed{P\{x\leq a\} = \int_a^{\infty} f_x(x)\,dx} \quad \text{--- (2)}$$

The above equation can be represented in terms of unit step function has.

$$\boxed{P\{x\leq a\} = \int_a^{\infty} f_x(x)\; u(x-a)\,dx} \quad \text{--- (3)}$$

eq (1) in eq (3)

$$P\{x\leq a\} = \int_a^{\infty} f_x(x)\, e^{v(x-a)}\,dx$$

$$= \int_a^{\infty} f_x(x)\, e^{vx} e^{-va}\,dx$$

$$P\{x\leq a\} = e^{-va}\int_a^{\infty} f_x(x)\, e^{vx}\,dx$$

$$= e^{-va}\left[\int_a^{\infty} f_x(x)\, e^{vx}\,dx\right]$$

$$P\{x\leq a\} = e^{-va} M_x(v)$$

note:- The maximum value of chernoff is called the bound.

chebyshev's inequality :-

for a random variable 'x' with a mean $\bar{x}$, and varience σx^2 then the chebyshev's inequality is given by

$$\boxed{P\{|x-\bar{x}| \leq a\} \leq \frac{\sigma x^2}{a^2}} \quad \text{--- (1)}$$

we know that

$$P\{x-\bar{x}\} \leq a\} = \int_{-\infty}^{\bar{x}-a} f_x(x)\,dx + \int_{\bar{x}+a}^{\infty} f_x(x)\,dx$$

$|x-\bar{x}| = a$

$x - \bar{x} = a$ | $-(x-\bar{x}) = a$

$x = a+\bar{x}$ | $-x+\bar{x} = a$

| $\bar{x}-a = x$

$$= \int_{|x-\bar{x}| \leq a}^{\infty} f_x(x)\,dx \quad \text{L.H.S}$$

we know that $\sigma x^2 = E[(x-\bar{x})^2] = \int_{-\infty}^{\infty} (x-\bar{x})^2 f_x(x)\,dx$

$$= (x-\bar{x})^2 \int_{-\infty}^{\infty} f_x(x)\,dx$$

$$= a^2 \int_{-\infty}^{\infty} f_x(x)\,dx \;= \text{R.H.S}$$

equating L.H.S & R.H.S

$$\int_{|x-\bar{x}| \leq a}^{\infty} f_x(x)\,dx \leq a^2 \int_{-\infty}^{\infty} f_x(x)\,dx$$

$$= a^2 \int_{|x-\bar{x}| \leq a}^{\infty} \cancel{f}_x(x)dx$$

$$= a^2 \left[P\{|x-\bar{x}| \leq a\} \right]$$

similarly probability of $P\{|x-\bar{x}| \geq a\} \leq 1 - \frac{\sigma_x^2}{a^2}$

i.e., Chebyshev's inequality

<u>Note</u>:- In the above equations, if $\sigma_x^2 \to 0$ and for any smaller value of a $(a \to 0)$ then the probability approches to it's mean value.

$$P\{|x-\bar{x}| \geq a\} = 1 - \frac{\sigma_x^2}{a^2}$$

$$P\{|x-\bar{x}| \to 0\} = 1 - \frac{0}{0}$$

$$P\{|x-\bar{x}| = 0\} = 1.$$

$$|x-\bar{x}| = 0$$

$$x = \bar{x}$$

$$P\{x|x = \bar{x}|\} = 1$$

<u>Markov's inequality condition</u>:-

Markov's inequality is applied to the non negative randomvariables and is given by

$$P\{x \geq a\} \leq \frac{E[x]}{a} \quad (a>0).$$

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Transformation of a random variable:-

Transformation is nothing but changing x. that is conversion of the given random variable, that is it is represented in-terms of another random variable 'y'.

$$\boxed{T[x] = Y}$$

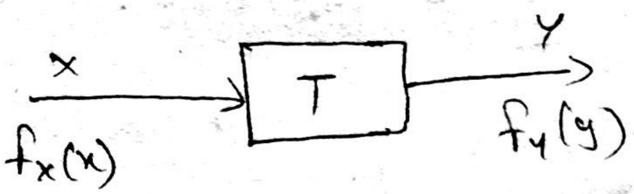
If the density function $f_x(x)$ and distribution function $F_x(x)$ of x are known, we can find the density function $f_Y(y)$ and distribution function $F_Y(y)$ depending on x and T and these are subclassified into three categories.

1. x and T both are continuous and either monotonically increasing or decreasing function.
2. x and T both are continuous but non-monotonic functions.
3. x is discrete and T is continuous.

Monotonic func transformations of continuous random variables

A transformation 'T' is called monotonically increasing if transformation of x.

$T(x_1) \le T(x_2)$ for any $x_1 \le x_2$ and it is monotonically decreasing function. If $T(x_1) \ge T(x_2)$ for any $x_1 < x_2$



Transformation of a variable x and Y.

i.e., $Y = T[x]$

for a particular of y_0 of Y it corresponds to a particular value of x_0 of x.

i.e., $T(x_0) = y_0$

The probability of event of $\{Y \le y\}$ must be equal to the probability of event $\{x \le x\}$

i.e., There is one to one correspondance b/w x & y

$$P\{x \le x\} = P\{Y \le y\}$$

$$F_X(x) = F_Y(y).$$

$$\int f_x(x)dx = \int f_y(y)dy$$

$$\int f_X(x)dx = \int f_Y(y)dy$$

$$\int f_X(x)\frac{dx}{dy} = \int f_Y(y)$$

$$\int f_X\{T[y_1]\frac{d}{dy}\cancel{(x)} = f_Y(y)$$

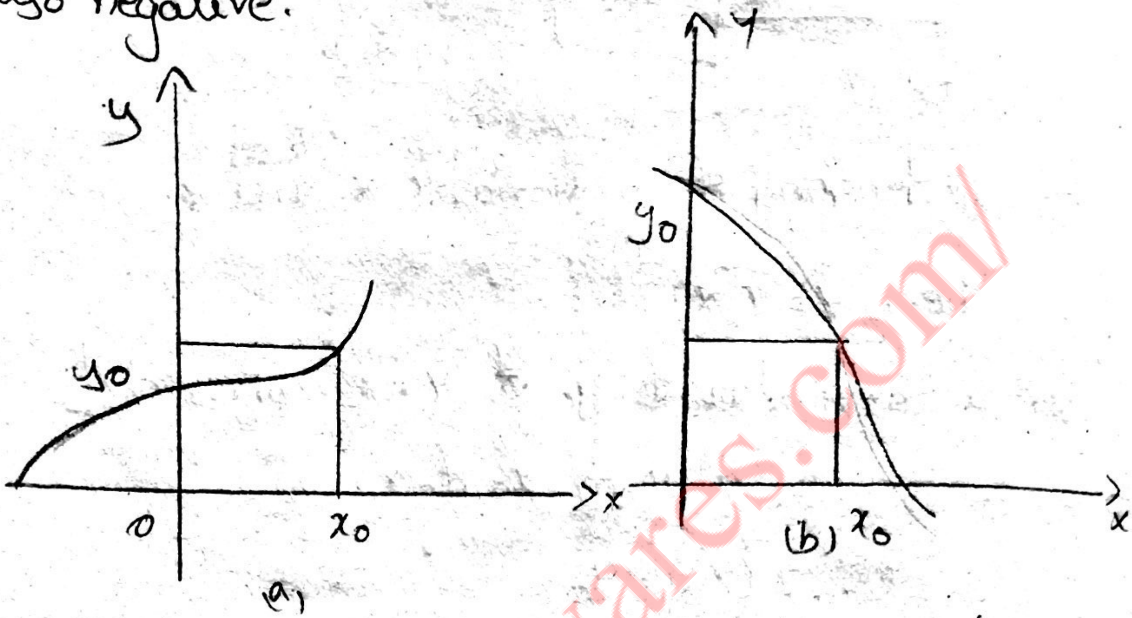
$$f_X(x)\frac{dx}{dy} = f_Y(y)$$

$$f_X(x\frac{dx}{dy}(T^{-1}(y)) = f_Y(y)$$

$$f_Y(y) = f_X(T^{-1}(y))\frac{d}{dy}T^{-1}(y)$$

$$\cancel{\boxed{\int f_x\left[T^{-1}(y)\right] \frac{d}{dy}\left(T^{-1}(y)\right) = f_Y(y)}}$$

Since, the slope of inverse transform of Y is also negative.



monotonic function

$$|f_Y(y)| = \left|f_X T^{-1}(y) \frac{d}{dy} T^{-1}(y)\right|$$

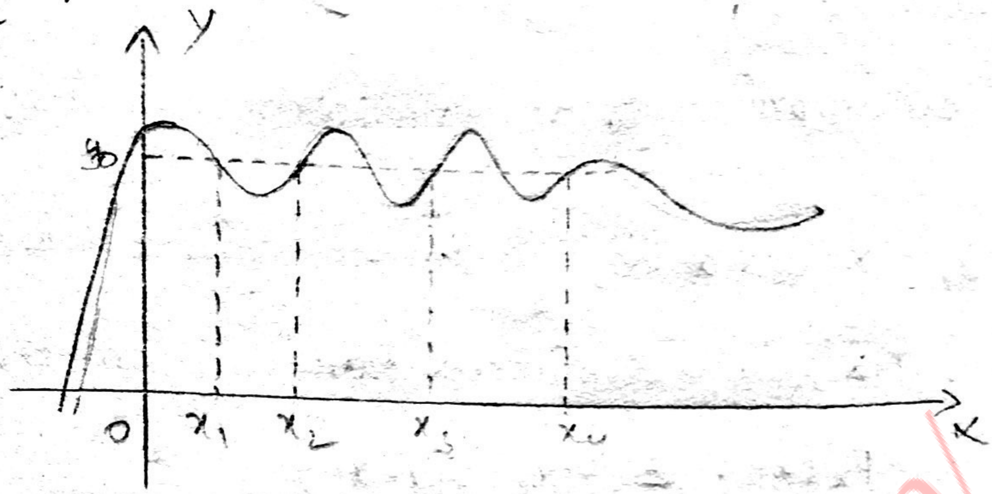
a) monotonically increasing function.

b) monotonically decreasing function.

i.e., $\left|\int f_X(T^{-1}(y)) \frac{d}{dy}(T^{-1}(y))\right| = |f_Y(y)|$

2. Non-monotonic function of a continuous random variable

non-monotonic function of a continuous random variable:



non-monotonic transformation functions

By observing the above graph, there are more than one time interval of x that corresponds to the event $\{Y \le y_0\}$

for the value of y_0 shown in the above figure. the event $\{y \le y_0\}$ corresponds to

$\{x \le x_1 \,\&\, x_2 \,\&\, x_3 \,\&\, x_4\}$

i.e., the probability of event $Y\{Y \le y_0\}$ must ~~be meet~~ be equal to the probability of event $\{x \le x_0\}$ i.e., $F_Y(y_0) = P\{Y \le y_0\} \Rightarrow P\{x|y \le y_0\}$

$$= \int_{\{x|y \le y_0\}} f_X(x)\, dy_0$$

$$F_Y(y_0) = \int_{(x|y \le y_0)} f_X(x)\, dx$$

$$\Rightarrow \int f_Y(y_0)\, dy_0 = \int_{x|y \le y_0} f_X(x)$$

$$f_Y(y_0) = \frac{d}{dy_0}\left[\int_{x|y \le y_0} f_X(x)\, dx\right]$$

$$f_Y(y_0) = \frac{d}{dy_0} \int_{(x|y \le y_0)} f_X(x)\, dx$$

3. Transformation of discreste random variables :-

If X is a discrete random variable while $Y = T[X]$ is a continuous transformation then

$$F_X(x) = \sum_{i=1}^{n} P(x_i)\, u(x - x_i)$$

$$f_X(x) = \sum_{i=1}^{n} P(x_i)\, \delta(x - x_i)$$

where the sum is taken to include all the possible values x_n ; $n = 1, 2, 3 \ldots\ldots \infty$

If the transformation is monotonic, there is one to one correspondance X and Y, so that a event $\{y_n\}$ corresponds to $\{x_n\}$

i.e., $F_X(y) = \sum_{i=1}^{n} P(y_i)\, u(y - y_i)$

$$F_Y(y) = \sum_{i=1}^{n} P(y_i)\, \delta(y - y_i)$$

where $y_n = T[x_n]$ and $P\{y_n\} = P\{x_n\}$

Problems:

① A random variable x is know to have a distribution function $F_X(x)$

$F_X(x) = u(x)\left[1 - e^{-x^2/b}\right]$ where $b > 0$ then find its density function

Given that

$$F_X(x) = u(x)\left[1 - e^{-x^2/b}\right]$$

WKT $f_X(x) = \frac{d}{dx} F_X(x)$

$$= \frac{d}{dx}\left[u(x)\left(1 - e^{-x^2/b}\right)\right]$$

$$= \frac{d}{dx}\left[u(x) - e^{-x^2/b}\, u(x)\right]$$

$$= \frac{d}{dx} u(x) - \frac{d}{dx} e^{-x^2/b}\, u(x)$$

$$= 0 - e^{-x^2/b} \frac{d}{dx}\left(-x^2/b\right)$$

$$= -e^{-x^2/b}\left(\frac{-2x}{b}\right).$$

$$\boxed{f_X(x) = \frac{2x}{b} e^{-x^2/b}}$$

② suppose a random proses is known to have a triangular probability density function by

$$f_X(x) = \begin{cases} 0 & 3 > x \geq 13 \\ \frac{x-3}{25} & 3 \leq x \leq 8 \\ 0.2 - \left(\frac{x-8}{25}\right) & 8 \leq x \leq 13. \end{cases}$$

Then find $P(x)$

that has values > 4.5 but not > 6.7

(a) $P[x] \Leftrightarrow P\{4.5 \le x \le 6.7\}$

$$\Rightarrow F_X(x) = \int_{4.5}^{6.7} f_x(x)\,dx$$

$$\Rightarrow \int_{4.5}^{6.7} \frac{x-3}{25}\,dx$$

$$= \frac{1}{25}\int_{4.5}^{6.7} (x-3)\,dx$$

$$= \frac{1}{25}\left\{\int_{4.5}^{6.7} x\,dx - \int_{4.5}^{6.7} 3\,dx\right\}$$

$$= \frac{1}{25}\left\{\left(\frac{x^2}{2}\right)_{4.5}^{6.7} - 3[x]_{4.5}^{6.7}\right\}$$

$$= \frac{1}{25}\left[\frac{(6.7)^2}{2} - \frac{(4.5)^2}{2} - 3[6.7 - 4.5]\right]$$

$$= \frac{1}{25}\left\{\frac{44.89}{2} - \frac{20.25}{2} - 3(2.2)\right\}$$

$$= \frac{1}{25}\{12.32 - 6.6\}$$

$$= \frac{1}{25}\{5.72\}$$

$$= 0.2288$$

$\therefore P\{4.5 \le x \le 6.7\} = 0.2288$

(3) find the [probability of the event] $P\{x \leq 5.5\}$ for a Gaussian Random variable having $a_x = 3$, $\sigma_x = 2$

(A) ~~Note~~: The distribution function $F_x(x)$ can be define interms of Gaussian Random variable has

$$F_x(x) = F_x\left(\frac{x - a_x}{\sigma_x}\right)$$

$$\Rightarrow F_x\left(\frac{5.5 - 3}{2}\right)$$

$$\Rightarrow F_x(1.25)$$

$\Rightarrow$ from the distribution table

$$F_x(1.25) = 0.8994$$

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(4) Assume that height of clouds above the ground at some location is the Gaussian Random variable 'x', with $a_x = 1830\,m$, $\sigma_x = 460\,m$ then find the probability that the clouds will be higher then 2750 m

(A) Given that $a_x = 1830\,m$

$\sigma_x = 400\,m$

$P\{x > 2750\,m\} = ?$

$x = 2750\,m$

$$F_X(x) = P\{X \le x\}$$

w.k.T

$$P\{X > x\} = 1 - P\{X \le x\}$$

$$F_X(x) = 1 - F_X(x)$$

$$= 1 - F_X\left(\frac{x - a_X}{\sigma_X}\right)$$

$$= 1 - F_X\left(\frac{2750 - 1830}{460}\right)$$

$$= 1 - F_X(2)$$

$$= 1 - 0.9773$$

$$P\{X > 2750\} = 0.0227$$

(5) The probability density function of a random variable x can be given as $f_X(x) = \begin{cases} x & , 0 < x < 1 \\ 2 - x & , 1 \le x \le 2 \\ 0 & , \text{else} \end{cases}$ then find

i, cumilative probability distribution function

2, find $P\{0.2 < x < 0.8\}$

3, find $P\{0.6 < x < 1.2\}$

Sol: (i) CPDF $\Rightarrow F_X(x)$

w.k.T $\quad f_X(x) = \frac{d}{dx} F_X(x)$

$$\Rightarrow \int_{-\infty}^{\infty} f_X(x)\,dx = F_X(x)$$

$$F_X(x) = \int_{-\infty}^{\infty} f_X(x)\,dx$$

$$= \int_{-\infty}^{0} f_X(x)dx + \int_{0}^{1} f_X(x)dx + \int_{1}^{2} f_X(x)dx + \int_{2}^{\infty} f_X(x)dx$$

$$= 0 + \int_{0}^{1} x\,dx + \int_{1}^{2} (2-x)dx + 0$$

$$= \int_{0}^{1} x\,dx + \int_{1}^{2} (2-x)\,dx$$

$$= \left(\frac{x^2}{2}\right)_0^1 + 2(x)_1^2 - \left(\frac{x^2}{2}\right)_1^2$$

$$= \frac{1}{2} + 2 - \frac{3}{2} = 1$$

$$\boxed{\text{CPDF } F_X(x) = 1}$$

ii, $P\{0.2 < X < 0.8\}$

w.k.T $P\{x_1 < X < x_2\} \Rightarrow \int_{x_1}^{x_2} f_X(x)dx$

$$P\{0.2 < X < 0.8\} = \int_{0.2}^{0.8} x\,dx$$

$$= \left(\frac{x^2}{2}\right)_{0.2}^{0.8}$$

$$= \left(\frac{0.64 - 0.04}{2}\right) = 0.3$$

$$P\{0.2 < X < 0.8\} = 0.3$$

iii, $P\{0.6 < X < 1.2\} = \int_{0.6}^{1.2} 2 - x\,dx$

$$= 2\int_{0.6}^{1.2} \cancel{x}dx \int_{0.6}^{1.2} x\,dx$$

$$2[x]_{0.6}^{1.2} - \left(\frac{x^2}{2}\right)_{0.6}^{1.2}$$

$$2[1.2-0.6] - \left(\frac{1.44-0.36}{2}\right)$$

$$2(0.6) - 0.54$$

$$= 0.66$$

(6) $\therefore P\{0.6 < X < 1.2\} = 0.66$

(6) A random variable x has the following probability function

x	-2	-1	0	1	2	3
P(x)	0.1	K	0.2	2K	0.3	K

Then find

1. find the value of K
2. Mean of x
3. varience of x

Sol

WKT

overall probability of any expression = 1

$$\sum_{i=1}^{N} P(x_i) = 1$$

$$\sum_{i=1}^{6} P(x_i) = 1$$

$$P(x_1)+P(x_2)+P(x_3)+P(x_4)+P(x_5)+P(x_6)=1$$

$$P(-2)+P(-1)+P(0)+P(1)+P(2)+P(3)=1$$

$$0.1+K+0.2+2K+0.3+K=1$$

$$0.6+4K=1$$

$$4K=1-0.6=0.4$$

$$K=\frac{0.4}{4}=0.1$$

$$\therefore \boxed{K=0.1}$$

(ii) Mean of X

w.k.T $E[x]=\sum_{i=1}^{N} x_i P(x_i)$

$$\Rightarrow \sum_{i=1}^{6} x_i P(x_i)$$

$$\Rightarrow x_1 P(x_1)+x_2 P(x_2)+x_3 P(x_3)+x_4 P(x_4)+x_5 P(x_5)+x_6 P(x_6)$$

$$=-2(0.1)+(-1)(0.1)+0(0.2)+1(0.2)+2(0.3)+3(0.1)$$

$$=0.8$$

$$\boxed{\bar{X}=0.8}$$

(iii) varience of X

w.k.T varience $\sigma_x^2=E[[X-\bar{X}]^2]=m_2-m_1^2$

where $m_2=\sum_{i=1}^{N} x_i^2 P(x_i)$

$\sum_{i=1}^{6} x_i^2 P(x_i)$

$x_1^2 P(x_1) + x_2^2 P(x_2) + x_3^2 P(x_3) + x_4^2 P(x_4) + x_5^2 P(x_5) +$
$x_6^2 P(x_6)$

$(-2)^2(0.1) + (-1)^2(0.1) + 0^2(0.2) + 1^2(0.2) + 2^2(0.3) + 3^2(0.1)$

$= 4(0.1) + 0.1 + 0.2 + 1.2 + 0.9$

$= 0.4 + 0.1 + 0.2 + 1.2 + 0.9$

$= 2.8$

$\therefore \sigma x^2 = m_2 - m_1^2$

$= 2.8 - (0.8)^2$

$= \cancel{2.8} = 2.8 - 0.64$

$= 2.16$

2) A random variable X has the following function

X	0	1	2	3	4	5	6	7	8
P(x)	a	3a	5a	7a	9a	11a	13a	15a	17a

1) find 'a' value

2) $P\{0 < x < 5\}$

3) what is the smallest 'x', $P\{X \le x\} > 0.5$

(1) $\sum_{i=1}^{N} P(x_i) = 1$

$\sum_{i=1}^{8} P(x_i) = 1$

$P(x_1) + P(x_2) + P(x_3) + P(x_4) + P(x_5) + P(x_6) + P(x_7) + P(x_8) + P(x_9)$

$P(0) + P(1) + P(2) + P(3) + P(4) + P(5) + P(6) + P(7) + P(8) = 1$

$a + 3a + 5a + 7a + 9a + 11a + 13a + 15a + 17a = 1$

$81a = 1$

$a = \frac{1}{81}$

$= 0.012$

2) $P\{0 < x \leq 5\}$

$\sum_{i=0}^{a} P(x_i)$

$\cancel{P(0)} + P(1) + P(2) + P(3) + P(4) \cancel{+ P(5)} \cancel{= 1}$

$\cancel{a} + 3a + 5a + 7a + 9a + \cancel{11a} \cancel{= 1}$

$\cancel{25a = 1}$

$\cancel{25(0.012)}$

$= 24a$

$= 24(0.012)$

$= 0.288$

iii) $P\{X \le x\} > 0.5$

if $N=1$ $\quad \sum_{i=1}^{N} P(x_i) \Rightarrow P(x_1)$

$$a = 0.012 \not> 0.5$$

if $N=2$ $\quad \sum_{N=1}^{N} P(x_i) = P(x_1)+P(x_2)$

$$= a+3a = 4a$$

$$= 4(0.012) = 0.048 \not> 0.5$$

if $N=3$ $\quad \sum_{N=1}^{N} P(x_i) = P(x_1)+P(x_2)+P(x_3)$

$$= a+3a+5a$$

$$= 9a = 9(0.012)$$

$$= 0.108 \not> 0.5$$

if $N=4$ $\quad \sum_{N=1}^{N} P(x_i) = P(x_1)+P(x_2)+P(x_3)+P(x_4)$

$$= a+3a+5a+7a$$

$$= 16a = 16(0.012)$$

$$= 0.192 \not> 0$$

if $N=5$ $\quad \sum_{N=1}^{N} P(x_i) = P(x_1)+P(x_2)+P(x_3)+P(x_4)+P(x_5)$

$$= a+3a+5a+7a+9a$$

$$= 25a = 25(0.012) = 0.3 \not> 0$$

if $N=6$ $\quad \sum_{N=1}^{N} P(x_i) = P(x_1)+P(x_2)+P(x_3)+P(x_4)+P(x_5)+P(x_6)$

$$= a+3a+5a+7a+9a+11a$$

$$= 36a = 0.432 \not> 0$$

if $\sum_{N=1}^{N} P(x_i) = P(x_1)+P(x_2)+P(x_3)+P(x_4)+P(x_5)+P(x_6)$
$N=7$
$+P(x_7)$

$= a+3a+5a+7a+9a+11a+13a$

$= 49a(0.012)$

$= 0.588$

hence the smallest value of X $P\{X\} \geq 0.5$ is

0.588 $\qquad P\{X \leq x\} > 0.5$

8) The PDF of random variable x is given by

$$f_x(x) = \begin{cases} K & a<x<b \\ 0 & \text{else} \end{cases}$$

then find i) K value

ii) let $a=1$ & $b=2$ and calculate

$P\{|x| \leq c\}$ where $c=0.5$

(A) i) K value

$$F_x(x) = \int f_x(x)\,dx$$

$$1 = \int_a^b K\,dx$$

$$1 = K(x)_a^b$$

$$\boxed{K = \frac{1}{b-a}}$$

ii) $P\{|x| \leq c\}$, $c=0.5$, $a=1$, $b=2$

$P\{-0.5 < x < 0.5\} = F_X(x)$

$$F_X(x) = \int f_X(x)\,dx$$

$$\int_a^b k\,dx \Rightarrow k(x)_a^b$$

$$= k[x]_1^2$$

$$= k[2-1]$$

$$= k$$

$$= \frac{1}{b-a}$$

$$= \frac{1}{2-1}$$

$$= 1$$

Let x be a continuous random variable with density function $f_X(x) = \begin{cases} \frac{x}{6}+K, & 0 \le x \le 3 \\ 0, & \text{else} \end{cases}$ then find

i, the value of k

ii, $P\{1 \le x \le 2\}$

Given that $f_X(x) = \begin{cases} \frac{x}{6}+K, & 0 \le x \le 3 \\ 0 & \text{else} \end{cases}$

$$F_X(x) = \int f_X(x)\,dx$$

$$1 = \int_0^3 \left(\frac{x}{6}+K\right)dx$$

$$1 = \int_0^3 \frac{x}{6}\,dx + \int_0^3 k\,dx$$

$$= \frac{1}{6}\left[\frac{x^2}{2}\right]_0^3 + k\left[x\right]_0^3$$

$$= \frac{1}{\cancel{6}_2}\left[\frac{\cancel{9}^3}{2}\right] + k[3]$$

$$1 = \frac{3}{4} + 3k$$

$$3k = 1 - \frac{3}{4}$$

$$3k = \frac{1}{4}$$

$$\boxed{k = \frac{1}{12}}$$

ii, $P\{1 \le x \le 2\} = F_X(x)$

$$F_X(x) = \int f_X(x)\,dx$$

$$= \int_1^2 \left(\frac{x}{6} + k\right)dx$$

$$= \int_1^2 \frac{x}{6}\,dx + \int_1^2 k\,dx$$

$$= \frac{1}{6}\left[\frac{x^2}{2}\right]_1^2 + k\left[x\right]_1^2$$

$$= \frac{1}{6}\left[\frac{\cancel{4}^2}{\cancel{2}_1} - \frac{1}{2}\right] + k[2-1]$$

$$= \frac{1}{\cancel{6}_2}\left[\frac{\cancel{3}^1}{2}\right] + k$$

$$= \frac{1}{4} + k = \frac{1}{4} + \frac{1}{12} = 0.333 = \frac{4}{12} = \frac{4}{12}$$

UNIT III
Multiple Random Variables and Operations on Multiple Random Variables

Multiple Random Variables:

- Joint Distribution Function and Properties
- Joint density Function and Properties
- Marginal Distribution and density Functions
- Conditional Distribution and density Functions
- Statistical Independence
- Distribution and density functions of Sum of Two Random Variables

Operations on Multiple Random Variables:

- Expected Value of a Function of Random Variables
- Joint Moments about the Origin
- Joint Central Moments
- Joint Characteristic Functions
- Jointly Gaussian Random Variables: Two Random Variables case

MULTIPLE RANDOM VARIABLES

Multiple Random Variables

In many applications we have to deal with more than two random variables. For example, in the navigation problem, the position of a space craft is represented by three random variables denoting the x, y and z coordinates. The noise affecting the R, G, B channels of color video may be represented by three random variables. In such situations, it is convenient to define the vector-valued random variables where each component of the vector is a random variable.

In this lecture, we extend the concepts of joint random variables to the case of multiple random variables. A generalized analysis will be presented for random variables defined on the same sample space.

Example1: Suppose we are interested in studying the height and weight of the students in a class. We can define the joint RV (X,Y) where X represents height and Y represents the weight.

Example 2 Suppose in a communication system X is the transmitted signal and Y is the corresponding noisy received signal. Then (X,Y) is a joint random variable.

Joint Probability Distribution Function:

Recall the definition of the distribution of a single random variable. The event $\{X \le x\}$ was used to define the probability distribution function $F_X(x)$. Given $F_X(x)$, we can find the probability of any event involving the random variable. Similarly, for two random variables X and Y, the event $\{X \le x, Y \le y\} = \{X \le x\} \cap \{Y \le y\}$ is considered as the representative event.

The probability $P\{X \le x, Y \le y\}\ \forall (x,y) \in \mathbb{R}^2$ is called the *joint distribution function of the random variables X and Y* and denoted by $F_{X,Y}(x,y)$.

Properties of Joint Probability Distribution Function:

The joint CDF satisfies the following properties:

1. $F_X(x)=F_{XY}(x,\infty)$, for any x (marginal CDF of X);
   Proof:
   $$\{X \le x\} = \{X \le x\} \cap \{Y \le +\infty\}$$
   $$\therefore F_X(x) = P(\{X \le x\}) = P(\{X \le x, Y \le \infty\}) = F_{XY}(x,+\infty)$$
   Similarly $F_Y(y) = F_{XY}(\infty, y)$.
2. $F_Y(y)=F_{XY}(\infty,y)$, for any y (marginal CDF of Y);
3. $F_{XY}(\infty,\infty)=1$;
4. $F_{XY}(-\infty,y)=F_{XY}(x,-\infty)=0$;
5. $P(x1<X\le x2,y1<Y\le y2)= F_{XY}(x2,y2)-F_{XY}(x1,y2)-F_{XY}(x2,y1)+F_{XY}(x1,y1)$;
6. if X and Y are independent, then $F_{XY}(x,y)=F_X(x)F_Y(y)$
7. $F_{X,Y}(x_1,y_1) \le F_{X,Y}(x_2,y_2)$ if $x_1 \le x_2$ and $y_1 \le y_2$
   Proof:

:

If $x_1 < x_2$ and $y_1 < y_2$,

$\{X \le x_1, Y \le y_1\} \subseteq \{X \le x_2, Y \le y_2\}$

$\therefore P\{X \le x_1, Y \le y_1\} \le P\{X \le x_2, Y \le y_2\}$

$\therefore F_{X,Y}(x_1, y_1) \le F_{X,Y}(x_2, y_2)$

Example1:

Consider two jointly distributed random variables X and Y with the joint CDF

$$F_{X,Y}(x,y) = \begin{cases} (1-e^{-2x})(1-e^{-y}) & x \ge 0, y \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal CDFs

(b) Find the probability $P\{1 < X \le 2, \ \ 1 < Y \le 2\}$

Solution:

(a)

$$F_X(x) = \lim_{y \to \infty} F_{X,Y}(x,y) = \begin{cases} 1-e^{-2x} & x \ge 0 \\ 0 & \text{elsewhere} \end{cases}$$

$$F_Y(y) = \lim_{x \to \infty} F_{X,Y}(x,y) = \begin{cases} 1-e^{-y} & y \ge 0 \\ 0 & \text{elsewhere} \end{cases}$$

(b)

$$\begin{aligned} P\{1 < X \le 2, \ \ 1 < Y \le 2\} &= F_{X,Y}(2,2) + F_{X,Y}(1,1) - F_{X,Y}(1,2) - F_{X,Y}(2,1) \\ &= (1-e^{-4})(1-e^{-2}) + (1-e^{-2})(1-e^{-1}) - (1-e^{-2})(1-e^{-2}) - (1-e^{-4})(1-e^{-1}) \\ &= 0.0272 \end{aligned}$$

Jointly distributed discrete random variables

If X and Y are two discrete random variables defined on the same probability space (S, F, P) such that X takes values from the countable subset R_X and Y takes values from the countable subset R_Y. Then the joint random variable (X, Y) can take values from the countable subset in $R_X \times R_Y$. The joint random variable (X, Y) is completely specified by their *joint probability mass function*

$$p_{X,Y}(x,y) = P\{s \mid X(s) = x, Y(s) = y\}, \ \ \forall (x,y) \in R_X \times R_Y.$$

Given $p_{X,Y}(x,y)$, we can determine other probabilities involving the random variables X and Y.

Remark

- $p_{X,Y}(x,y) = 0$ for $(x,y) \notin R_X \times R_Y$
- $\sum\sum_{(x,y) \in R_X \times R_Y} p_{X,Y}(x,y) = 1$

This is because

$$\begin{aligned} \sum\sum_{(x,y) \in R_X \times R_Y} p_{X,Y}(x,y) &= P\Big(\bigcup_{(x,y) \in R_X \times R_Y} \{x,y\}\Big) \\ &= P(R_X \times R_Y) \\ &= P\{s \mid (X(s), Y(s)) \in (R_X \times R_Y)\} \\ &= P(S) = 1 \end{aligned}$$

- ***Marginal Probability Mass Functions:*** The probability mass functions $p_X(x)$ and $p_Y(y)$ are obtained from the joint probability mass function as follows

$$p_X(x) = P\{X = x\} \cup R_Y$$
$$= \sum_{y \in R_Y} p_{X,Y}(x, y)$$

and similarly

$$p_Y(y) = \sum_{x \in R_X} p_{X,Y}(x, y)$$

These probability mass functions $p_X(x)$ and $p_Y(y)$ obtained from the joint probability mass functions are called *marginal probability mass functions*.

Example Consider the random variables X and Y with the joint probability mass function as tabulated in Table . The marginal probabilities are as shown in the last column and the last row

X \ Y	0	1	2	$p_Y(y)$
0	0.25	0.1	0.15	0.5
1	0.14	0.35	0.01	0.5
$p_X(x)$	0.39	0.45		

Joint Probability Density Function

If X and Y are two continuous random variables and their joint distribution function is continuous in both x and y, then we can define *joint probability density function* $f_{X,Y}(x, y)$ by

$$f_{X,Y}(x, y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x, y), \text{ provided it exists.}$$

Clearly $F_{X,Y}(x, y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{X,Y}(u, v)\,dv\,du$

Properties of Joint Probability Density Function:

- $f_{X,Y}(x, y)$ is always a non-negative quantity. That is,

$$f_{X,Y}(x, y) \geq 0 \quad \forall (x, y) \in \mathbb{R}^2$$

- $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X,Y}(x, y)\,dx\,dy = 1$
- Marginal probability density functions can be defined as

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, y)\,dy, \quad \text{for all } x,$$
$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x, y)\,dx, \quad \text{for all } y.$$

- The probability of any Borel set B can be obtained by

$$P(B) = \iint_{(x,y) \in B} f_{X,Y}(x, y)\,dx\,dy$$

Marginal Distribution and density Functions:

The probability distribution functions of random variables X and Y obtained from joint distribution function is called ad marginal distribution functions. i.e.

$F_X(x)=F_{XY}(x,\infty)$, for any x (marginal CDF of X);

Proof:

$$\{X \le x\} = \{X \le x\} \cap \{Y \le +\infty\}$$

$$\therefore F_X(x) = P(\{X \le x\}) = P(\{X \le x, Y \le \infty\}) = F_{XY}(x, +\infty)$$

Similarly $F_Y(y) = F_{XY}(\infty, y)$.

The marginal density functions $f_X(x)$ and $f_Y(y)$ of two joint RVs X and Y are given by the derivatives of the corresponding marginal distribution functions. Thus

$$\begin{aligned} f_X(x) &= \tfrac{d}{dx} F_X(x) \\ &= \tfrac{d}{dx} F_X(x,\infty) \\ &= \tfrac{d}{dx} \int_{-\infty}^{x} \left(\int_{-\infty}^{\infty} f_{X,Y}(u,y)dy \right) du \\ &= \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy \end{aligned}$$

$$\text{and similarly} \quad f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

The marginal CDF and pdf are same as the CDF and pdf of the concerned single random variable. The *marginal* term simply refers that it is derived from the corresponding joint distribution or density function of two or more jointly random variables.

Example2: The joint density function $f_{X,Y}(x,y)$ in the previous example is

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y) \\ &= \frac{\partial^2}{\partial x \partial y}[(1-e^{-2x})(1-e^{-y})] \quad x \ge 0, y \ge 0 \\ &= 2e^{-2x}e^{-y} \quad x \ge 0, y \ge 0 \end{aligned}$$

Example3: The joint pdf of two random variables X and Y are given by

$$\begin{aligned} f_{X,Y}(x,y) &= cxy \quad 0 \le x \le 2,\ 0 \le y \le 2 \\ &= 0 \quad \text{otherwise} \end{aligned}$$

(i) Find c.

(ii) Find $F_{X,y}(x,y)$

(iii) Find $f_X(x)$ and $f_Y(y)$.

(iv) What is the probability $P(0 < X \le 1, 0 < Y \le 1)$?

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)dydx = c\int_0^2\int_0^2 xydydx$$

$$\therefore c = \frac{1}{4}$$

$$F_{X,Y}(x,y) = \frac{1}{4}\int_0^y\int_0^x uvdudv$$

$$= \frac{x^2y^2}{16}$$

$$f_X(x) = \int_0^2 \frac{xy}{4}dy \quad 0 \le y \le 2$$

$$= \frac{x}{2} \quad 0 \le y \le 2$$

Similarly

$$f_Y(y) = \frac{y}{2} \qquad 0 \le y \le 2$$

$$P(0 < X \le 1, 0 < Y \le 1)$$

$$= F_{X,Y}(1,1) + F_{X,Y}(0,0) - F_{X,Y}(0,1) - F_{X,Y}(1,0)$$

$$= \frac{1}{16} + 0 - 0 - 0$$

$$= \frac{1}{16}$$

Conditional Distribution and Density functions

We discussed conditional probability in an earlier lecture. For two events A and B with $P(B) \neq 0$, the conditional probability $P(A/B)$ was defined as

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

Clearly, the conditional probability can be defined on events involving a random variable *X*.

Conditional distribution function

Consider the event $\{X \le x\}$ and any event *B* involving the random variable *X*. The conditional distribution function of *X* given *B* is defined as

$$F_X(x/B) = P[\{X \le x\}/B]$$

$$= \frac{P[\{X \le x\} \cap B]}{P(B)} \qquad P(B) \neq 0$$

Properties of Conditional distribution function

We can verify that $F_X(x/B)$ satisfies all the properties of the distribution function. Particularly.

- $F_X(-\infty/B) = 0$ and $F_X(\infty/B) = 1$.
- $0 \le F_X(x/B) \le 1$

- $F_X(x/B)$ is a non-decreasing function of x.
- $$P(\{x_1 < X \le x_2\}/B) = P(\{X \le x_2\}/B) - P(\{X \le x_1\}/B)$$
  $$= F_X(x_2/B) - F_X(x_1/B)$$

Conditional density function

In a similar manner, we can define the conditional density function $f_X(x/B)$ of the random variable X given the event B as

$$f_X(x/B) = \frac{d}{dx}F_X(x/B)$$

Properties of Conditional density function:

All the properties of the pdf applies to the conditional pdf and we can easily show that

- $f_X(x/B) \ge 0$
- $\int_{-\infty}^{\infty} f_X(x/B)dx = F_X(\infty/B) = 1$
- $F_X(x/B) = \int_{-\infty}^{x} f_X(u/B)du$
- $$P(\{x_1 < X \le x_2\}/B) = F_X(x_2/B) - F_X(x_1/B)$$
  $$= \int_{x_1}^{x_2} f_X(x/B)dx$$

Let (X, Y) be a discrete bivariate random vector with joint pmf f(x, y) and marginal pmfs $f_X(x)$ and $f_Y(y)$. For any x such that $P(X = x) = f_X(x) > 0$, the conditional pmf of Y given that X = x is the function of y denoted by f(y|x) and defined by

$$f(y|x) = P(Y = y|X = x) = \frac{f(x,y)}{f_X(x)}.$$

For any y such that $P(Y = y) = f_Y(y) > 0$, the conditional pmf of X given that Y = y is the function of x denoted by f(x|y) and defined by

$$f(x|y) = P(X = x|Y = y) = \frac{f(x,y)}{f_Y(y)}.$$

Example 1: Suppose X is a random variable with the distribution function $F_X(x)$. Define $B = \{X \le b\}$.

Then

$$F_X(x/B) = \frac{P(\{X \le x\} \cap B)}{P(B)}$$
$$= \frac{P(\{X \le x\} \cap \{X \le b\})}{P\{X \le b\}}$$
$$= \frac{P(\{X \le x\} \cap \{X \le b\})}{F_X(b)}$$

Case 1: $x<b$

Then

$$F_X(x/B)=\frac{P(\{X\le x\}\cap\{X\le b\})}{F_X(b)}$$
$$=\frac{P(\{X\le x\})}{F_X(b)}=\frac{F_X(x)}{F_X(b)}$$

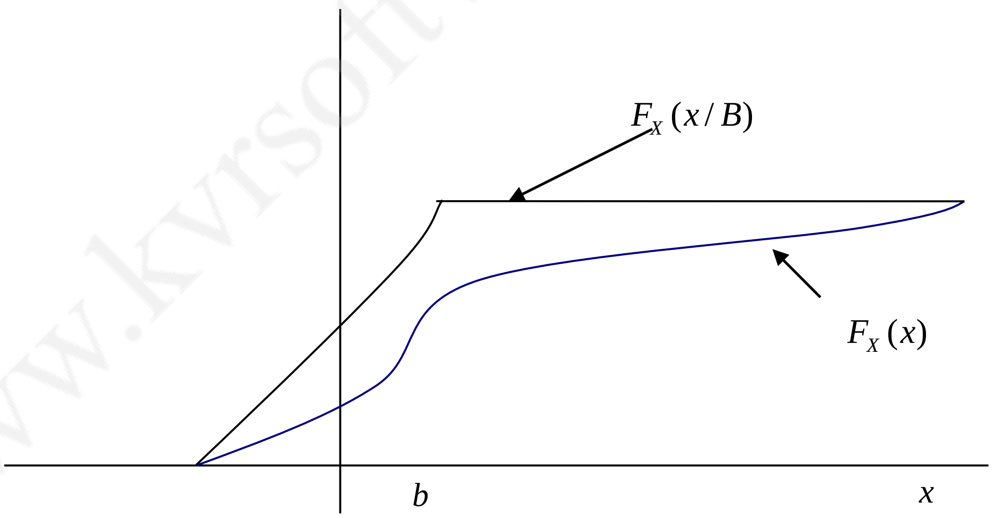
And $f_X(x/B)=\frac{d}{dx}\frac{F_X(x)}{F_X(b)}=\frac{f_X(x)}{f_X(b)}$

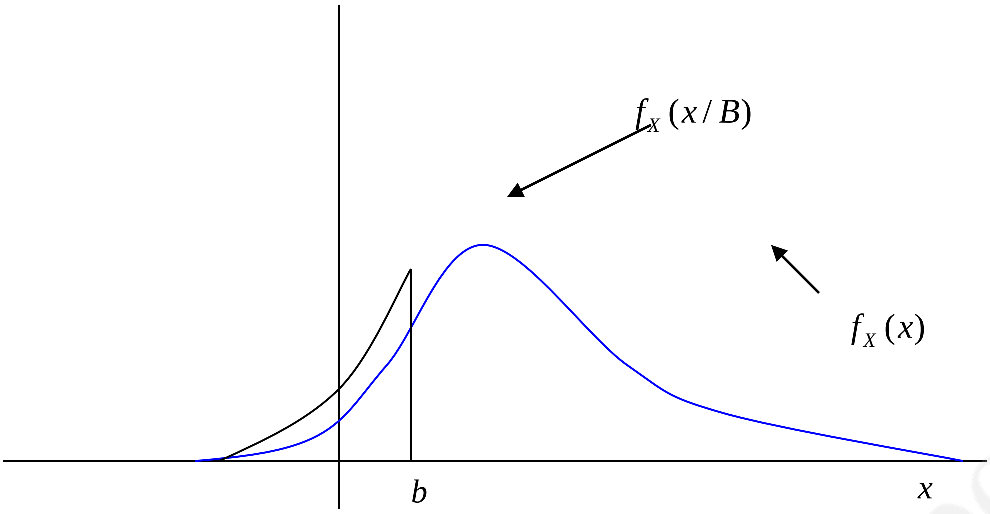
Case 2: $x\ge b$

$$F_X(x/B)=\frac{P(\{X\le x\}\cap\{X\le b\})}{F_X(b)}$$
$$=\frac{P(\{X\le x\})}{F_X(b)}=\frac{F_X(b)}{F_X(b)}=1$$

and $f_X(x/B)=\frac{d}{dx}F_X(x/B)=0$

$F_X(x/B)$ and $f_X(x/B)$ are plotted in the following figures.





Example 2 Suppose X is a random variable with the distribution function $F_X(x)$ and $B = \{X > b\}$.

Then

$$F_X(x/B) = \frac{P(\{X \le x\} \cap B)}{P(B)}$$

$$= \frac{P(\{X \le x\} \cap \{X > b\})}{P\{X > b\}}$$

$$= \frac{P(\{X \le x\} \cap \{X > b\})}{1 - F_X(b)}$$

For $x \le b$, $\{X \le x\} \cap \{X > b\} = \phi$. Therefore,

$$F_X(x/B) = 0 \qquad x \le b$$

For $x > b$, $\{X \le x\} \cap \{X > b\} = \{b < X \le x\}$ Therefore,

$$F_X(x/B) = \frac{P(\{b < X \le x\})}{1 - F_X(b)}$$

$$= \frac{F_X(x) - F_X(b)}{1 - F_X(b)}$$

Thus,

$$F_X(x/B) = \begin{cases} 0 & x \le b \\ \dfrac{F_X(x) - F_X(b)}{1 - F_X(b)} & \text{otherwise} \end{cases}$$

The corresponding pdf is given by

$$f_X(x/B)=\begin{cases}0 & x\le b\\ \dfrac{f_X(x)}{1-F_X(b)} & \text{otherwise}\end{cases}$$

Example 3 The joint density of X and Y is given by

$$f(x,y)=\begin{cases}\frac{15}{2}x(2-x-y) & 0<x<1, 0<y<1\\ 0 & \text{otherwise}\end{cases}.$$

Compute the condition density of X, given that $Y=y$, where $0<y<1$.

Solution For $0<x<1, 0<y<1$, we have

$$f_X(x|y)=\frac{f(x,y)}{f_Y(y)}=\frac{f(x,y)}{\int_{-\infty}^{\infty}f(x,y)\,dx}$$
$$=\frac{x(2-x-y)}{\int_0^1 x(2-x-y)\,dx}=\frac{x(2-x-y)}{\frac{2}{3}-\frac{y}{2}}=\frac{6x(2-x-y)}{4-3y}.$$

Example4:

Let the continuous random vector (X, Y) have joint pdf

$$f(x, y) = e^{-y}, \quad 0 < x < y < \infty.$$

The marginal of X is

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_{x}^{\infty} e^{-y}dy = e6{-}x.$$

Thus, marginally, X has an exponential distribution. The conditional distribution of Y is

$$f(y|x) = \frac{f(x, y)}{f_X(x)} = \begin{cases} \frac{e^{-y}}{e^{-x}} = e^{-(y-x)}, & \text{if } y > x, \\ \frac{0}{e^{-x}} = 0, & \text{if } y \leq x \end{cases}$$

Conditional Probability Distribution Function

Consider two continuous jointly random variables X and Y with the joint probability distribution function $F_{X,Y}(x,y)$. We are interested to find the conditional distribution function of one of the random variables on the condition of a particular value of the other random variable.

We *cannot* define the conditional distribution function of the random variable Y on the condition of the event $\{X = x\}$ by the relation

$$F_{Y/X}(y/x) = P(Y \le y / X = x)$$
$$= \frac{P(Y \le y, X = x)}{P(X = x)}$$

as $P(X = x) = 0$ in the above expression. The conditional distribution function is defined in the *limiting sense* as follows:

$$F_{Y/X}(y/x) = lim_{\Delta x \to 0} P(Y \le y / x < X \le x + \Delta x)$$
$$= lim_{\Delta x \to 0} \frac{P(Y \le y, x < X \le x + \Delta x)}{P(x < X \le x + \Delta x)}$$
$$= lim_{\Delta x \to 0} \frac{\int_{\infty}^{y} f_{X,Y}(x,u)\Delta x du}{f_X(x)\Delta x}$$
$$= \frac{\int_{\infty}^{y} f_{X,Y}(x,u)du}{f_X(x)}$$

$$\therefore F_{Y/X}(y/x) = \frac{\int_{\infty}^{y} f_{X,Y}(x,u)du}{f_X(x)}$$

Conditional Probability Density Function

$f_{Y/X}(y / X = x) = f_{Y/X}(y/x)$ is called the *conditional probability density function* of Y given X

Let us define the conditional distribution function .

The conditional density is defined in the limiting sense as follows

$f_{Y/X}(y/X=x)=\lim_{\Delta y\to 0}(F_{Y/X}(y+\Delta y/X=x)-F_{Y/X}(y/X=x))/\Delta y$

$\therefore f_{Y/X}(y/X=x)=\lim_{\Delta y\to 0,\Delta x\to 0}(F_{Y/X}(y+\Delta y/x<X\le x+\Delta x)-F_{Y/X}(y/x<X\le x+\Delta x))/\Delta y$

Because, $(X=x)=\lim_{\Delta x\to 0}(x<X\le x+\Delta x)$

The right hand side of the highlighted equation is

$$\begin{aligned}
&\lim_{\Delta y\to 0,\Delta x\to 0}(F_{Y/X}(y+\Delta y/x<X<x+\Delta x)-F_{Y/X}(y/x<X<x+\Delta x))/\Delta y\\
&=\lim_{\Delta y\to 0,\Delta x\to 0}(P(y<Y\le y+\Delta y/x<X\le x+\Delta x))/\Delta y\\
&=\lim_{\Delta y\to 0,\Delta x\to 0}(P(y<Y\le y+\Delta y,x<X\le x+\Delta x))/P(x<X\le x+\Delta x)\Delta y\\
&=\lim_{\Delta y\to 0,\Delta x\to 0}f_{X,Y}(x,y)\Delta x\Delta y/f_X(x)\Delta x\Delta y\\
&=f_{X,Y}(x,y)/f_X(x)
\end{aligned}$$

$$\therefore f_{Y/X}(y/x)=f_{X,Y}(x,y)/f_X(x)$$

Similarly we have

$$\therefore f_{X/Y}(x/y)=f_{X,Y}(x,y)/f_Y(y)$$

Two random variables are *statistically independent* if for all $(x,y)\in\mathbb{R}^2$,

$f_{Y/X}(y/x)=f_Y(y)$

or equivalently

$f_{X,Y}(x,y)=f_X(x)f_Y(y)$

-

Example 2 X and Y are two jointly random variables with the joint pdf given by

$$\begin{aligned}f_{X,Y}(x,y)&=k\text{ for }0\le x\le 1\\&=0\text{ otherwise}\end{aligned}$$

find,

(a) k

(b) $f_X(x)$ and $f_Y(y)$

(a) $f_{X/Y}(x/y)$

Solution:

Since $\int\int_{-\infty}^{\infty} f_{X,Y}(x,y)dydx = 1$

We get

$$k\, x \frac{1}{2} x 1 x 1 = 1$$
$$\Rightarrow k = 2$$

$$\therefore f_{X,Y}(x,y) = 2 \quad for\ 0 \le x \le 1 \text{ as } y \le x$$
$$= 0 \text{ otherwise}$$

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy = 2\int_0^x dy = 2x$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx = 2\int_y^1 dx = 2(1-y)$$

Independent Random Variables (or) Statistical Independence

Let and be two random variables characterized by the joint distribution function

X Y

$$F_{X,Y}(x,y) = P\{X \le x, Y \le y\}$$

and the corresponding joint density function $f_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y)$

Then X and Y are independent if $\forall (x,y) \in \mathbb{R}^2$, $\{X \le x\}$ and $\{Y \le y\}$ are independent events. Thus,

$$F_{X,Y}(x,y) = P\{X \le x, Y \le y\}$$
$$= P\{X \le x\}P\{Y \le y\}$$
$$= F_X(x)F_Y(y)$$
$$\therefore f_{X,Y}(x,y) = \frac{\partial^2 F_{X,Y}(x,y)}{\partial x \partial y}$$
$$= \frac{dF_X(x)}{dx}\frac{dF_Y(y)}{dy}$$
$$= f_X(x)f_Y(y)$$
$$\therefore f_{X,Y}(x,y) = f_X(x)f_Y(y)$$

and equivalently $f_{Y/X}(y) = f_Y(y)$

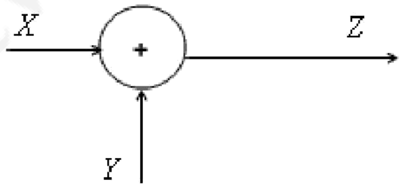
Density function of Sum of Two Random Variables:

We are often interested in finding out the probability density function of a function of two or more RVs. Following are a few examples.

- The received signal by a communication receiver is given by

$$Z = X + Y$$

where Z is received signal which is the superposition of the message signal X and the noise Y.



- The frequently applied operations on communication signals like modulation, demodulation, correlation etc. involve multiplication of two signals in the form $Z = XY$.

We have to know about the probability distribution of Z in any analysis of Z. More formally, given two random variables X and Y with joint probability density function $f_{X,Y}(x,y)$ and a function $Z = g(X,Y)$, we have to find $f_Z(z)$.

In this lecture, we shall address this problem.

We consider the transformation $g:\mathbb{R}^2 \to \mathbb{R}$.

Consider the event $\{Z \le z\}$ corresponding to each z. We can find a variable subset $D_Z \subseteq \mathbb{R}^2$ such that $D_Z = \{(x,y) \mid g(x,y) \le z\}$.

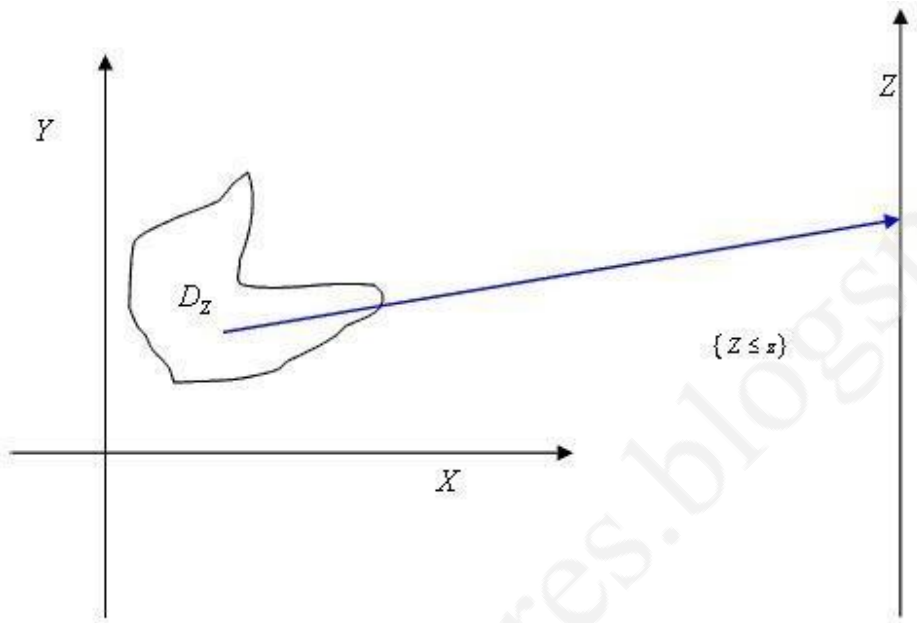


Figure 1

$$\begin{aligned}\therefore F_Z(z) &= P(\{Z \le z\})\\ &= P\{(x,y) \mid (x,y) \in D_z\}\\ &= \iint_{(x,y)\in D_z} f_{X,Y}(x,y)\,dydx\end{aligned}$$

$$\text{and } f_Z(z) = \frac{dF_Z(z)}{dz}$$

Probability density function of $Z = X + Y$.

Consider Figure 2

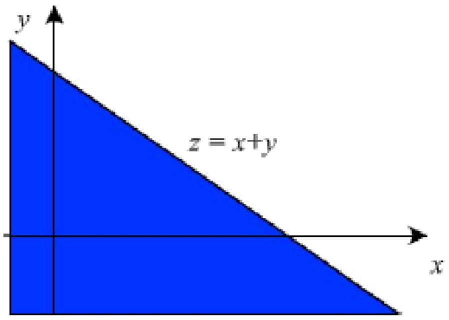


Figure 2

We have

$Z \le z$

$\Rightarrow X + Y \le z$

Therefore, D_Z is the colored region in the Figure

$$\therefore F_Z(z) = \iint_{(x,y)\in D_z} f_{X,Y}(x,y)\,dx\,dy$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{z-x} f_{X,Y}(x,y)\,dy\right]dx$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{z} f_{X,Y}(x,u-x)\,du\right]dx \quad \text{substituting } y = u - x$$

$$= \int_{-\infty}^{z}\left[\int_{-\infty}^{\infty} f_{X,Y}(x,u-x)\,dx\right]du \quad \text{interchanging the order of integration}$$

$$\therefore f_Z(z) = \frac{d}{dz}\int_{-\infty}^{z}\left[\int_{-\infty}^{\infty} f_{X,Y}(x,u-x)\,dx\right]du$$

$$= \int_{-\infty}^{\infty} f_{X,Y}(x,u-x)\,dx$$

$$\therefore f_Z(z) = \int_{-\infty}^{\infty} f_{X,Y}(x,u-x)\,dx$$

OPERATIONS ON MULTIPLE RANDOM VARIABLES

Expected Values of Functions of Random Variables

Introduction:

In this Part of Unit we will see the concepts of expectation such as mean, variance, moments, characteristic function, Moment generating function on Multiple Random variables. We are already familiar with same operations on Single Random variable. This can be used as basic for our topics we are going to see on multiple random variables.

Function of joint random variables:

If g(x,y) is a function of two random variables X and Y with joint density function $f_{x,y}(x,y)$ then the expected value of the function g(x,y) is given as

$$\overline{g} = E\,[g(x,y)] \text{ or}$$

$$\overline{g} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dx\,dy$$

Similarly, for N Random variables $X_1, X_2, \ldots X_N$ With joint density function $f_{x1,x2,\ldots Xn}(x_1,x_2,\ldots x_n)$, the expected value of the function $g(x_1,x_2,\ldots x_n)$ is given as

$$\overline{g} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} g(x_1,x_2,\ldots x_N)\, f_{x_1,x_2,\ldots x_N}(x_1,x_2,\ldots x_N)\; dx_1 dx_2 \ldots dx_N$$

Properties :

The properties of E(X) for continuous random variables are the same as for discrete ones:

1. If X and Y are random variables on a sample space Ω then E(X + Y) = E(X) + E(Y). (linearity I)
2. If a and b are constants then E(aX + b) = aE(X) + b.

If $Y = g(X)$ is a function of a discrete random variable X, then $EY = Eg(X) = \sum_{x \in R_X} g(x) p_X(x)$

Suppose $Z = g(X,Y)$ is a function of continuous random variables X and Y then the expected value of Z is given by

$$EZ = Eg(X,y) = \int_{-\infty}^{\infty} z f_Z(z) dz$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y) dxdy$$

Thus EZ can be computed without explicitly determining $f_Z(z)$.

We can establish the above result as follows.

Suppose $Z = g(X,Y)$ has n roots $(x_i, y_i),\ i = 1, 2, .., n$ at $Z = z$. Then

$$\{z < Z \leq z + \Delta z\} = \bigcup_{i=1}^{n} \{(x_i, y_i) \in \Delta D_i\}$$

Where

ΔD_i Is the differential region containing (x_i, y_i). The mapping is illustrated in Figure 1 for $n = 3$.

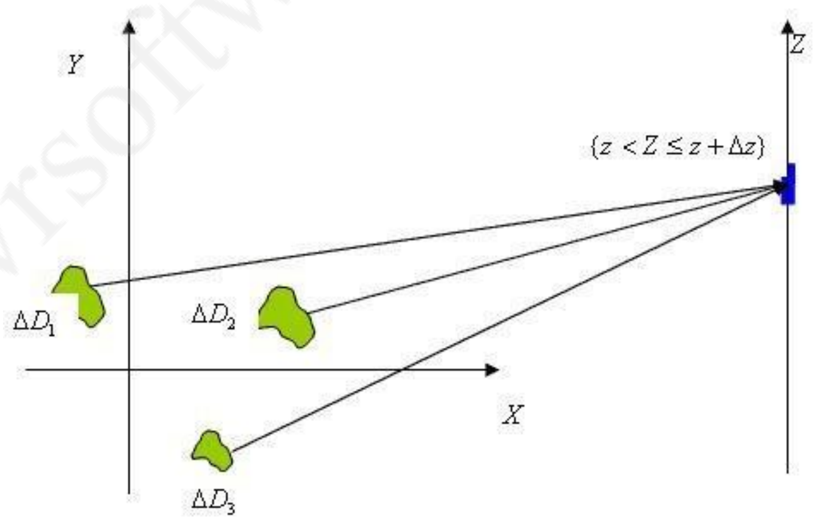


Figure 1

Note that

$$P(\{z < Z \le z + \Delta z\}) = f_Z(z)\Delta z = \sum_{(x_i, y_i) \in D_i} f_{X,Y}(x_i, y_i)\Delta x_i \Delta y_i$$

$$\therefore z f_Z(z)\Delta z = \sum_{(x_i, y_i) \in \Delta D_i} z f_{X,Y}(x_i, y_i)\Delta x_i \Delta y_i$$

$$= \sum_{(x_i, y_i) \in \Delta D_i} g(x_i, y_i) f_{X,Y}(x_i, y_i)\Delta x_i \Delta y_i$$

As Z is varied over the entire Z axis, the corresponding (non-overlapping) differential regions in $X-Y$ plane cover the entire plane.

$$\therefore \int_{-\infty}^{\infty} z f_Z(z)dz = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)dxdy$$

Thus,

$$\boxed{Eg(X,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)dxdy}$$

If $Z = g(X,Y)$ is a function of discrete random variables X and Y, we can similarly show that

$$EZ = Eg(X,Y) = \sum_{x,y \in R_X \times R_Y} \sum g(x,y) p_{X,Y}(x,y)$$

Example 1 The joint pdf of two random variables X and Y is given by

$$f_{X,Y}(x,y) = \frac{1}{4}xy \quad 0 \le x \le 2,\ 0 \le y \le 2$$
$$= 0 \quad \text{otherwise}$$

Find the joint expectation of $g(X,Y) = X^2Y$

$$
\begin{aligned}
Eg(X,Y) &= EX^2Y \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f_{X,Y}(x,y)\,dxdy \\
&= \int_0^2\int_0^2 x^2 y \frac{1}{4} xy\,dxdy \\
&= \frac{1}{4}\int_0^2 x^3 dx \int_0^2 y^2 dy \\
&= \frac{1}{4} \times \frac{2^4}{4} \times \frac{2^3}{3} \\
&= \frac{8}{3}
\end{aligned}
$$

Example 2 If $Z = aX + bY$, where a and b are constants, then

$$EZ = aEX + bEY$$

Proof:

$$
\begin{aligned}
EZ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (ax+by) f_{X,Y}(x,y)\,dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} ax f_{X,Y}(x,y)\,dxdy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} by f_{X,Y}(x,y)\,dxdy \\
&= \int_{-\infty}^{\infty} ax \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dydx + \int_{-\infty}^{\infty} by \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dxdy \\
&= a\int_{-\infty}^{\infty} x f_X(x)\,dx + b\int_{-\infty}^{\infty} y f_Y(y)\,dy \\
&= aEX + bEY
\end{aligned}
$$

Thus, expectation is a linear operator.

Example 3

Consider the discrete random variables X and Y discussed in Example 4 in lecture 18.The joint probability mass function of the random variables are tabulated in Table . Find the joint expectation of $g(X,Y) = XY$.

X / Y	0	1	2	$p_Y(y)$
0	0.25	0.1	0.15	0.5
1	0.14	0.35	0.01	0.5
$p_X(x)$	0.39	0.45	0.16	

$$\text{Clearly, } EXY = \sum_{x,y \in R_X \times R_Y} \sum g(x,y) p_{X,Y}(x,y)$$
$$= 1 \times 1 \times 0.35 + 1 \times 2 \times 0.01$$
$$= 0.37$$

Remark

(1) We have earlier shown that expectation is a linear operator. We can generally write

$$E[a_1 g_1(X,Y) + a_2 g_2(X,Y)] = a_1 E g_1(X,Y) + a_2 E g_2(X,Y)$$

Thus $$E(XY + 5\log_e XY) = EXY + 5E\log_e XY$$

(2) If X and Y are independent random variables and $g(X,Y) = g_1(X)g_2(Y)$,then

$$Eg(X,Y) = Eg_1(X)g_2(Y)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(X)g_2(Y) f_{X,Y}(x,y)dx$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g_1(X)g_2(Y) f_X(x) f_Y(y)dxdy$$
$$= \int_{-\infty}^{\infty} g_1(X) f_X(x)dx \int_{-\infty}^{\infty} g_2(Y) f_Y(y)dy$$
$$= Eg_1(X)Eg_2(Y)$$

Joint Moments of Random Variables

Just like the moments of a random variable provide a summary description of the random variable, so also the *joint moments* provide summary description of two random variables. For two continuous random variables X and Y , *the joint moment of order* $m+n$ is defined as

$$E(X^m Y^n) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x^m y^n f_{X,Y}(x,y)dxdy$$

And the joint central moment of order $m+n$ is defined as

$$E(X-\mu_X)^m(Y-\mu_Y)^n = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)^m(y-\mu_Y)^n f_{X,Y}(x,y)dxdy$$

where $\mu_X = EX$ and $\mu_Y = EY$

Remark

(1) If X and Y are discrete random variables, the joint expectation of order m and n is defined as

$$E(X^mY^n) = \sum_{(x,y)\in R_{X,Y}}\sum x^m y^n p_{X,Y}(x,y)$$

$$E(X-\mu_X)^m(Y-\mu_Y)^n = \sum_{(x,y)\in R_{X,Y}}\sum (x-\mu_X)^m(y-\mu_Y)^n p_{X,Y}(x,y)$$

(2) If $m=1$ and $n=1$, we have the second-order moment of the random variables X and Y given by

$$E(XY) = \begin{cases} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{X,Y}(x,y)dxdy & \text{if } X \text{ and } Y \text{ are continuous} \\ \sum_{(x,y)\in R_{X,Y}}\sum xyp_{X,Y}(x,y) & \text{if } X \text{ and } Y \text{ are discrete} \end{cases}$$

(3) If X and Y are independent, $E(XY) = EXEY$

Covariance of two random variables

The *covariance* of two random variables X and Y is defined as

$$Cov(X,Y) = E(X-\mu_X)(Y-\mu_Y)$$

Cov(X, Y) is also denoted as $\sigma_{X,Y}$.

Expanding the right-hand side, we get

$$\begin{aligned} Cov(X,Y) &= E(X-\mu_X)(Y-\mu_Y) \\ &= E(XY-\mu_Y X-\mu_X Y+\mu_X\mu_Y) \\ &= EXY-\mu_Y EX-\mu_X EY+\mu_X\mu_Y \\ &= EXY-\mu_X\mu_Y \end{aligned}$$

The ratio $\rho(X,Y) = \dfrac{Cov(X,Y)}{\sigma_X\sigma_Y}$ is called the **correlation coefficient**.

If $\rho_{X,Y} > 0$ then X and Y are called positively correlated.

If $\rho_{X,Y} < 0$ then X and Y are called negatively correlated

If $\rho_{X,Y} = 0$ then X and Y are uncorrelated.

We will also show that $|\rho(X,Y)| \leq 1.$ To establish the relation, we prove the following result:

For two random variables X and Y $E^2(XY) \leq EX^2 EY^2$

Proof:

Consider the random variable $Z = aX + Y$

$$E(aX + Y)^2 \geq 0$$
$$\Rightarrow a^2 EX^2 + EY^2 + 2aEXY \geq 0.$$

Non-negativity of the left-hand side implies that its minimum also must be nonnegative.

For the minimum value,

$$\frac{dEZ^2}{da} = 0 \Rightarrow a = -\frac{EXY}{EX^2}$$

so the corresponding minimum is

$$\frac{E^2XY}{EX^2} + EY^2 - 2\frac{E^2XY}{EX^2}$$
$$= EY^2 - \frac{E^2XY}{EX^2}$$

Since the minimum is nonnegative,

$$EY^2 - \frac{E^2XY}{EX^2} \geq 0$$
$$\Rightarrow E^2XY \leq EX^2 EY^2$$
$$\Rightarrow |EXY| \leq \sqrt{EX^2}\sqrt{EY^2}$$

Now

$$\begin{aligned}\rho(X,Y) &= \frac{Cov(X,Y)}{\sigma_X\sigma_Y}\\ &= \frac{E(X-\mu_X)(Y-\mu_Y)}{\sqrt{E(X-\mu_X)^2E(Y-\mu_Y)^2}}\\ \therefore |\rho(X,Y)| &= \frac{|E(X-\mu_X)(Y-\mu_Y)|}{\sqrt{E(X-\mu_X)^2E(Y-\mu_Y)^2}}\\ &\le \frac{\sqrt{E(X-\mu_X)^2}\sqrt{E(Y-\mu_Y)^2}}{\sqrt{E(X-\mu_X)^2E(Y-\mu_Y)^2}}\\ &= 1\end{aligned}$$

Thus $|\rho(X,Y)| \le 1$

Uncorrelated random variables

Two random variables X and Y are called *uncorrelated* if

$$Cov(X,Y) = 0$$

which also means

$$E(XY) = \mu_X\mu_Y$$

Recall that if X and Y are independent random variables, then $f_{X,Y}(x,y) = f_X(x)f_Y(y)$.

$$\begin{aligned} EXY &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_{X,Y}(x,y)dxdy \qquad \text{assuming } X \text{ and } Y \text{ are continuous}\\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf_X(x)f_Y(y)dxdy\\ &= \int_{-\infty}^{\infty} xf_X(x)dx\int_{-\infty}^{\infty} yf_Y(y)dy\end{aligned}$$

then $\quad = EXEY$

Thus two independent random variables are always uncorrelated.

Note that independence implies uncorrelated. But uncorrelated generally does not imply independence (except for jointly Gaussian random variables).

Joint Characteristic Functions of Two Random Variables

The *joint characteristic function* of two random variables X and Y is defined by

$$\phi_{X,Y}(\omega_1,\omega_2) = Ee^{j\omega_1 x + j\omega_2 y}$$

If X and Y are jointly continuous random variables, then

$$\phi_{X,Y}(\omega_1,\omega_2)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f_{X,Y}(x,y)e^{j\omega_1x+j\omega_2y}dydx$$

Note that $\phi_{X,Y}(\omega_1,\omega_2)$ is same as the two-dimensional Fourier transform with the basis function $e^{j\omega_1x+j\omega_2y}$ instead of $e^{-(j\omega_1x+j\omega_2y)}$.

$f_{X,Y}(x,y)$ is related to the joint characteristic function by the Fourier inversion formula

$$f_{X,Y}(x,y)=\frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi_{X,Y}(\omega_1,\omega_2)e^{-j\omega_1x-j\omega_2y}d\omega_1d\omega_2$$

If X and Y are discrete random variables, we can define the joint characteristic function in terms of the joint probability mass function as follows:

$$\phi_{X,Y}(\omega_1,\omega_2)=\sum_{(x,y)\in R_X\times R_Y}\sum p_{X,Y}(x,y)e^{j\omega_1x+j\omega_2y}$$

Properties of the Joint Characteristic Function

The joint characteristic function has properties similar to the properties of the chacteristic function of a single random variable. We can easily establish the following properties:

1. $\phi_X(\omega)=\phi_{X,Y}(\omega,0)$
2. $\phi_Y(\omega)=\phi_{X,Y}(0,\omega)$
3. If X and Y are independent random variables, then

$$\begin{aligned}\phi_{X,Y}(\omega_1,\omega_2)&=Ee^{j\omega_1X+j\omega_2Y}\\&=E(e^{j\omega_1X}e^{j\omega_2Y})\\&=Ee^{j\omega_1X}Ee^{j\omega_2Y}\\&=\phi_X(\omega_1)\phi_Y(\omega_2)\end{aligned}$$

4. We have,

$$\phi_{X,Y}(\omega_1,\omega_2) = Ee^{j\omega_1 X + j\omega_2 Y}$$

$$= E(1 + j\omega_1 X + j\omega_2 Y + \frac{j^2(\omega_1 X + j\omega_2 Y)^2}{2} + \ldots\ldots\ldots\ldots)$$

$$= 1 + j\omega_1 EX + j\omega_2 EY + \frac{j^2\omega_1^2 EX^2}{2} + \frac{j^2\omega_2^2 EY^2}{2} + \omega_1\omega_2 EXY + \ldots..$$

Hence,

$$\phi_{X,Y}(0,0) = 1$$

$$EX = \frac{1}{j}\frac{\partial}{\partial\omega_1}\phi_{X,Y}(\omega_1,\omega_2)\Big|_{\omega_1=0}$$

$$EY = \frac{1}{j}\frac{\partial}{\partial\omega_2}\phi_{X,Y}(\omega_1,\omega_2)\Big|_{\omega_2=0}$$

$$EXY = \frac{1}{j^2}\frac{\partial^2\phi_{X,Y}(\omega_1,\omega_2)}{\partial\omega_1\partial\omega_2}\Big|_{\omega_1=0,\omega_2=0}$$

In general, the $(m+n)th$ order joint moment is given by

$$EX^mY^n = \frac{1}{j^{m+n}}\frac{\partial^m\partial^n\phi_{X,Y}(\omega_1,\omega_2)}{\partial\omega_1^m\partial\omega_2^n}\Big|_{\omega_1=0,\omega_2=0}$$

Example 2 The joint characteristic function of the jointly Gaussian random variables X and Y with the joint pdf

$$f_{X,Y}(x,y) = \frac{e^{-\frac{1}{2(1-\rho_{X,Y}^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho_{X,Y}\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]}}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}}$$

Let us recall the characteristic function of a Gaussian random variable

$$X \sim N(\mu_X, \sigma_X^2)$$

$$\phi_X(\omega) = Ee^{j\omega X}$$

$$= \frac{1}{\sqrt{2\pi}\sigma_X} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2} . e^{j\omega x} dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma_X} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{x^2-2(\mu_X-\sigma_X^2 j\omega)x+(\mu_X-\sigma_X^2 j\omega)^2-(\mu_X-\sigma_X^2 j\omega)^2+\mu_X^2}{\sigma_X^2}} dx$$

$$= e^{\frac{1}{2}\frac{(-\sigma_X^2\omega^2+2\mu_X\sigma_X^2 j\omega)}{\sigma_X^2}} \underbrace{\frac{1}{\sqrt{2\pi}\sigma_X} \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu_X-\sigma_X^2 j\omega}{\sigma_X}\right)^2} dx}_{\text{Area under a Gaussian}}$$

$$= e^{\mu_X j\omega - \sigma_X^2\omega^2/2} \times 1$$

$$= e^{\mu_X j\omega - \sigma_X^{4\,2}\omega^2/2}$$

If X and Y is jointly Gaussian,

$$f_{X,Y}(x,y) = \frac{e^{-\frac{1}{2(1-\rho_{X,Y}^2)}\left[\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho_{X,y}\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right)+\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2\right]}}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}}$$

we can similarly show that

$$\phi_{X,Y}(\omega_1,\omega_2) = Ee^{j(X\omega_1+Y\omega_2)}$$
$$= e^{j\mu_X\omega_1 + j\mu_Y\omega_2 - \frac{1}{2}(\sigma_X^2\omega_1^2 + 2\rho_{X,Y}\sigma_X\sigma_Y\omega_1\omega_2 + \sigma_Y^2\omega_2^2)}$$

We can use the joint characteristic functions to simplify the probabilistic analysis as illustrated on next page:

Jointly Gaussian Random Variables

Many practically occurring random variables are modeled as jointly Gaussian random variables. For example, noise samples at different instants in the communication system are modeled as *jointly Gaussian random variables*.

Two random variables X and Y are called jointly Gaussian if their joint probability density

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}} e^{-\frac{1}{2(1-\rho_{X,Y}^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} - 2\rho_{XY}\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]}, \quad -\infty < x < \infty,\ -\infty < y < \infty$$

The joint pdf is determined by 5 parameters

- means μ_X and μ_Y
- variances σ_X^2 and σ_Y^2
- correlation coefficient $\rho_{X,Y}$

We denote the jointly Gaussian random variables X and Y with these parameters as

$$(X,Y) \sim N(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho_{X,Y})$$

The joint pdf has a bell shape centered at (μ_X, μ_Y) as shown in the Figure 1 below. The variances σ_X^2 and σ_Y^2 determine the spread of the pdf surface and $\rho_{X,Y}$ determines the orientation of the surface in the $X-Y$ plane.

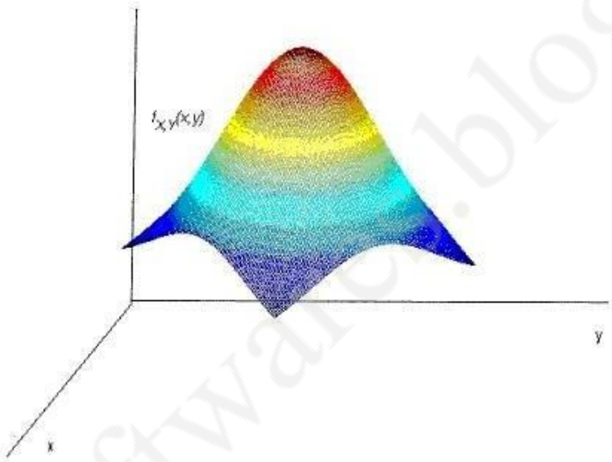


Figure 1 Jointly Gaussian PDF surface

Properties of jointly Gaussian random variables

(1) If X and Y are jointly Gaussian, then X and Y are both Gaussian.

We have

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$

$$= \int_{-\infty}^{\infty} \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}} e^{-\frac{1}{2(1-\rho_{X,Y}^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} - 2\rho_{XY}\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]} dy$$

$$= \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}}{\sqrt{2\pi}\sigma_X} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho_{X,Y}^2}} e^{-\frac{1}{2(1-\rho_{X,Y}^2)}\left[\frac{\rho_{X,Y}^2(x-\mu_X)^2}{\sigma_X^2} - 2\rho_{XY}\frac{(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]} dy$$

$$= \frac{e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}}{\sqrt{2\pi}\sigma_X} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_Y\sqrt{1-\rho_{X,Y}^2}} e^{-\frac{1}{2\sigma_Y^2(1-\rho_{X,Y}^2)}\left[(y-\mu_Y - \frac{\rho_{X,Y}\sigma_Y(x-\mu_X)}{\sigma_X}\right]^2} dy$$

$$= \frac{1}{\sqrt{2\pi}\sigma_X} e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}$$

Similarly

$$f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma_Y} e^{-\frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2}$$

(2) The converse of the above result is not true. If each of X and Y is Gaussian, X and Y are not necessarily jointly Gaussian. Suppose

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y} e^{-\frac{1}{2}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]}(1 + \sin x \sin y)$$

$f_{X,Y}(x,y)$ in this example is non-Gaussian and qualifies to be a joint pdf. Because,

$f_{X,Y}(x,y) \geq 0$ And

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi\sigma_X\sigma_Y}e^{-\frac{1}{2}\left[\frac{(x-\mu_X)^2}{\sigma_X^2}+\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]}(1+\sin x\sin y)dydx$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi\sigma_X\sigma_Y}e^{-\frac{1}{2}\left[\frac{(x-\mu_X)^2}{\sigma_X^2}+\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]}dydx+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi\sigma_X\sigma_Y}e^{-\frac{1}{2}\left[\frac{(x-\mu_X)^2}{\sigma_X^2}+\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]}\sin x\sin y\,dydx$$

$$=1+\frac{1}{2\pi\sigma_X\sigma_Y}\int_{-\infty}^{\infty}e^{-\frac{1}{2}\frac{(x-\mu_X)^2}{\sigma_X^2}}\sin x\,dx\;\underbrace{\int_{-\infty}^{\infty}e^{-\frac{1}{2}\frac{(y-\mu_Y)^2}{\sigma_Y^2}}\sin y\;dy}_{\text{integration of an odd function}}$$

$$=1+0$$

$$=1$$

The marginal density $f_X(x)$ is given by

$$f_X(x)=\int_{-\infty}^{\infty}\frac{1}{2\pi\sigma_X\sigma_Y}e^{-\frac{1}{2}\left[\frac{(x-\mu_X)^2}{\sigma_X^2}+\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]}(1+\sin x\sin y)dy$$

$$=\int_{-\infty}^{\infty}\frac{1}{2\pi\sigma_X\sigma_Y}e^{-\frac{1}{2}\left[\frac{(x-\mu_X)^2}{\sigma_X^2}+\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]}dy+\int_{-\infty}^{\infty}\frac{1}{2\pi\sigma_X\sigma_Y}\underbrace{e^{-\frac{1}{2(1-\rho_{X,Y}^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2}+\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]}\sin x\sin y\,dy}_{\text{integration of an odd function}}$$

$$=\frac{1}{\sqrt{2\pi}\sigma_X}e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}+0$$

$$=\frac{1}{\sqrt{2\pi}\sigma_X}e^{-\frac{1}{2}\left(\frac{x-\mu_X}{\sigma_X}\right)^2}$$

Similarly, $f_Y(y)=\frac{1}{\sqrt{2\pi}\sigma_Y}e^{-\frac{1}{2}\left(\frac{y-\mu_Y}{\sigma_Y}\right)^2}$

Thus X and Y are both Gaussian, but not jointly Gaussian.

(3) If X and Y are jointly Gaussian, then for any constants a and b ,the random variable Z given by $Z=aX+bY$ is Gaussian with mean $\mu_Z=a\mu_X+b\mu_Y$ and variance $\sigma_Z^{\ 2}=a^2\sigma_X^{\ 2}+b^2\sigma_Y^{\ 2}+2ab\sigma_X\sigma_Y\rho_{X,Y}$

(4) Two jointly Gaussian RVs X and Y are independent if and only if X and Y are uncorrelated $(\rho_{X,Y}=0)$ .Observe that if X and Y are uncorrelated, then

$$
\begin{aligned}
f_{X,Y}(x,y) &= \frac{1}{2\pi\sigma_X\sigma_Y} e^{-\frac{1}{2}\left[\frac{(x-\mu_X)^2}{\sigma_X^2}+\frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]} \\
&= \frac{1}{\sqrt{2\pi}\sigma_X} e^{\frac{(x-\mu_X)^2}{\sigma_X^2}} \frac{1}{\sqrt{2\pi}\sigma_Y} e^{\frac{(y-\mu_Y)^2}{\sigma_Y^2}} \\
&= f_X(x) f_Y(y)
\end{aligned}
$$

Example 1 Suppose *X* and *Y* are two jointly-Gaussian 0-mean random variables with variances of 1 and 4 respectively and a covariance of 1. Find the joint PDF $f_{X,Y}(x,y)$

$$\mu_X = \mu_Y = 0,\ \sigma_X^2 = 1,\ \sigma_Y^2 = 4 \text{ and } \operatorname{cov}(X,Y) = 1.$$

$$\therefore \rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y} = \frac{1}{1\times 2} = \frac{1}{2}$$

and

$$f_{X,Y}(x,y) = \frac{1}{2\pi\times 1\times 2\sqrt{1-\frac{1}{4}}} e^{-\frac{1}{2\times\frac{3}{4}}\left[\frac{x^2}{1} - 2\times\frac{1}{2}\times\frac{xy}{1\times 2} + \frac{y^2}{4}\right]}$$

We have

$$= \frac{1}{2\sqrt{3}\pi} e^{-\frac{2}{3}\left[x^2 - \frac{xy}{2} + \frac{y^2}{4}\right]}$$

Example 2 Linear transformation of two random variables

Suppose $Z = aX + bY$ then

$$\phi_Z(\omega) = Ee^{j\omega Z} = Ee^{j(aX+bY)\omega} = \phi_{X,Y}(a\omega, b\omega)$$

If X and Y are jointly Gaussian, then

$$
\begin{aligned}
\phi_Z(\omega) &= \phi_{X,Y}(a\omega, b\omega) \\
&= e^{j(\mu_X+\mu_Y)\omega - \frac{1}{2}(a^2\sigma_X^2 + 2\rho_{X,Y}ab\sigma_X\sigma_Y + b^2\sigma_Y^2)\omega^2}
\end{aligned}
$$

Which is the characteristic function of a Gaussian random variable with mean $\mu_Z = \mu_X + \mu_Y$ and variance $\sigma_Z^2 = \sigma_X^2 + 2\rho_{X,Y}\sigma_X\sigma_Y + \sigma_Y^2$

thus the linear transformation of two Gaussian random variables is a Gaussian random variable

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unit - IV

Random processors - temporal characteristics

Random process concept: A random variable 'x' is a function of the possible out comes of 'x' of an experiment.

i.e.. the random process now becomes as a function of both s and t.

In other words we can say that a time function x(t, s) to every out come 's'.

the family of such functions denoted as x(s, t) is called a random process.

where x is denoted as specific value of 'x' random variables

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classifications of processors:-

It is convinent to classified random processors according to the characteristics of 't' and random variable 'x'.

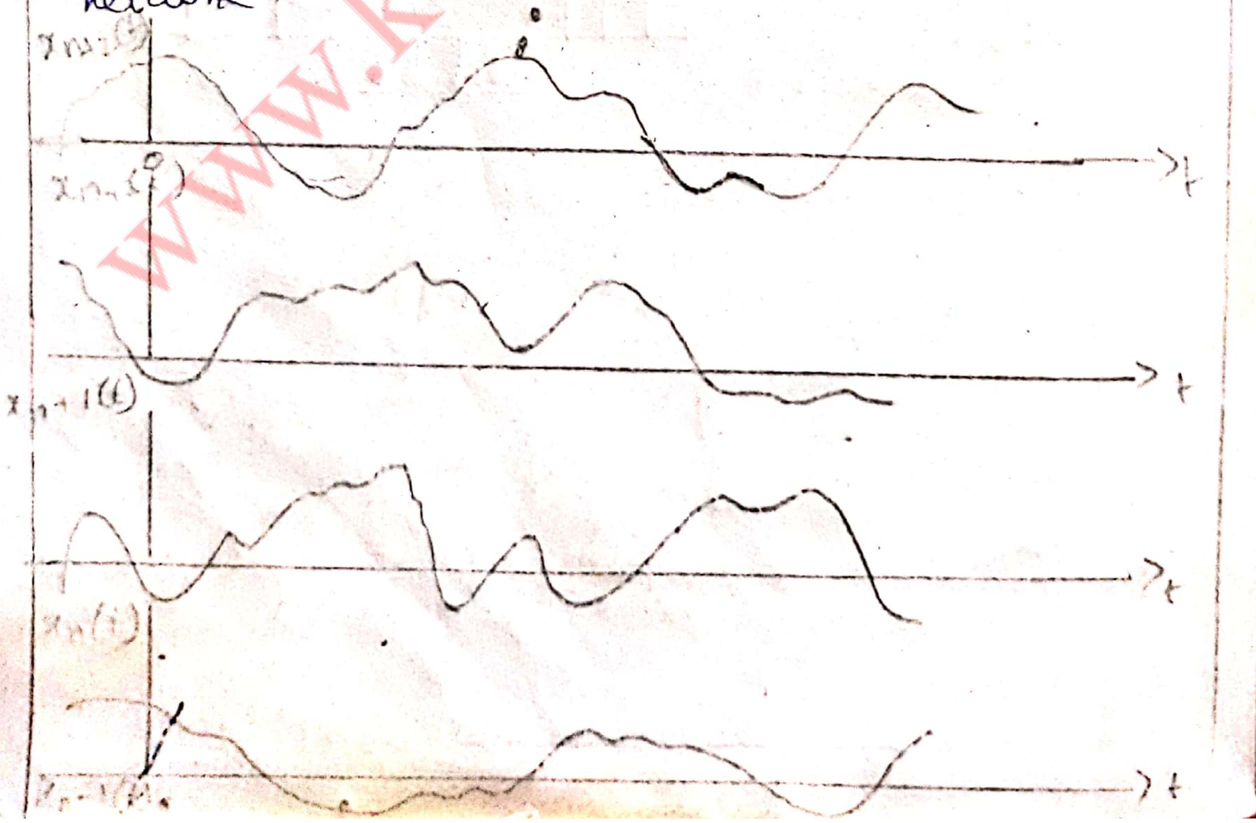
i.e., $x = x(t)$ at a time 't'.

So based on the values of x and t we shall classify these having ranges $-\infty < x < \infty$ and $-\infty < t < \infty$

1. continuous random process:-

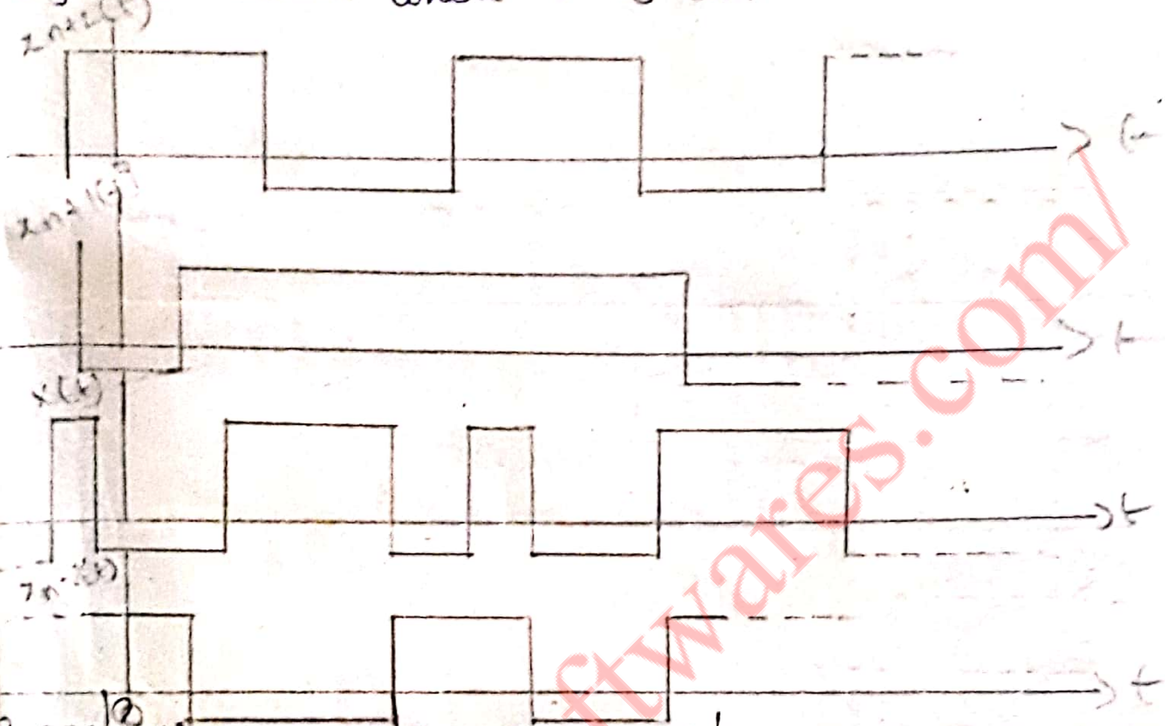
If x is continuous and t can have any of continuous values then $x(t)$ is called a continuous random process.

Eg:- Thermal noise generated by any ~~va~~ reliable network



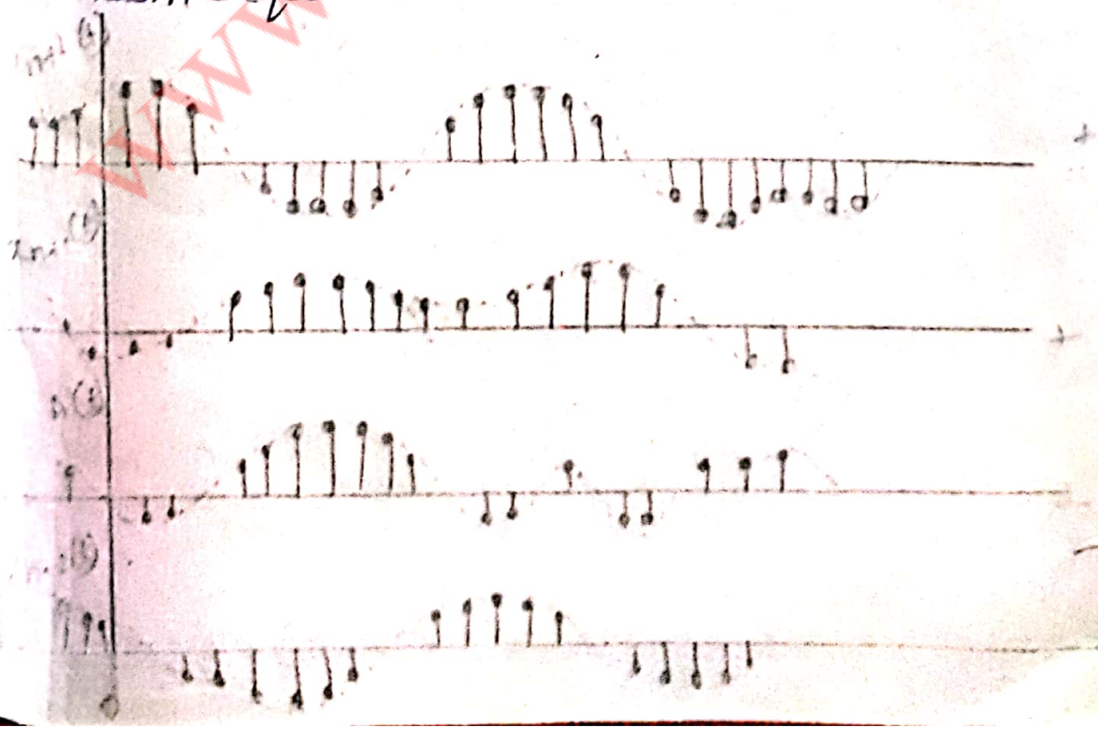
2. Discrete random process:-

A Discrete random process is a one that corresponds to a random variable 'x'. having only discrete value while 't' is continuous.



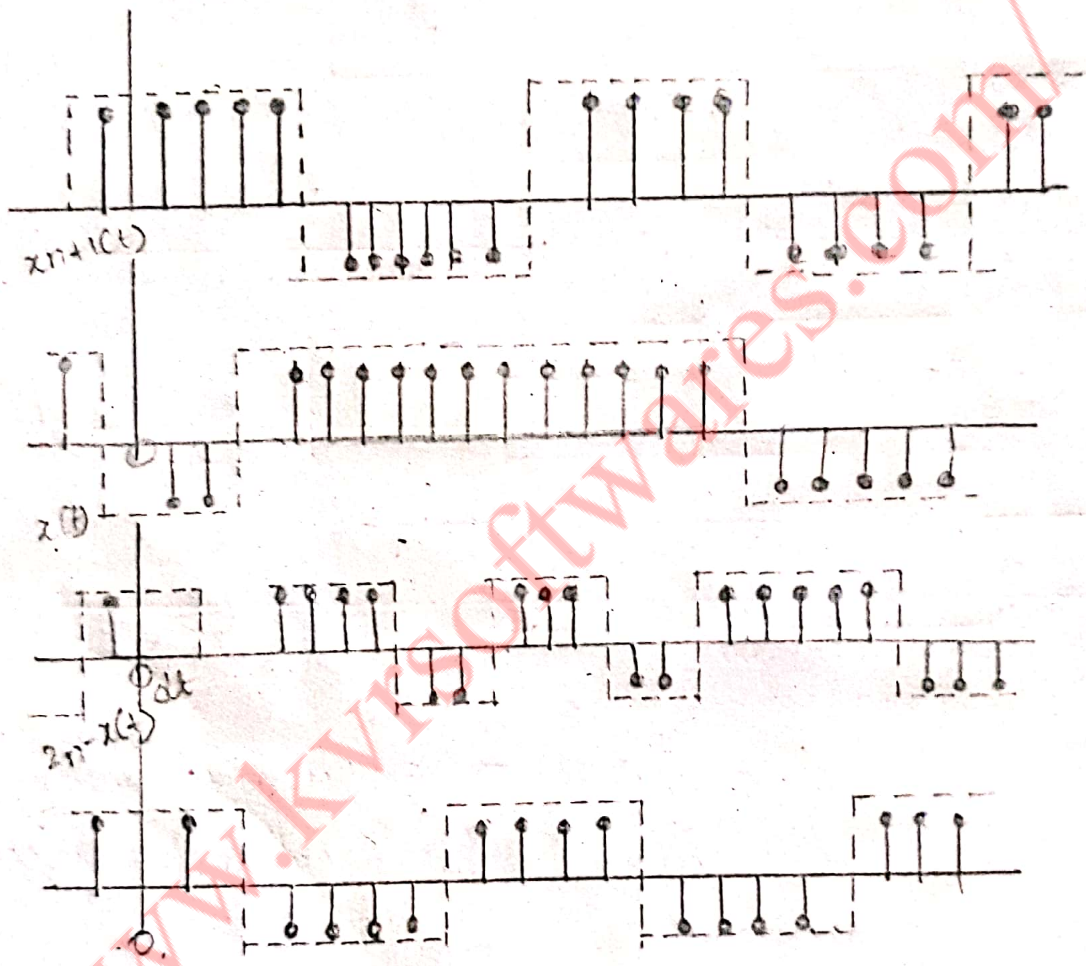
3. continuous random sequence:-

A random process for which x is continuous and 't' has only discrete value is called continuous random sequence.



4) Discrete random sequence:-

A random process for which x is discrete and 't' has discrete value is called discrete random sequence.



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stationary and independence (statistical independence)

A random process becomes a random variable when time fixed i.e., at some perticular time

$\therefore x = X(t)$

The random variable will possess statistical properties such as mean, varience, Moments that corresponds to it's density function.

* If two or more random variables are involve, all these processes should be done at two time instances (for two variables) and this can be extented for N dimensional Joint density function.

A random process is said to be stationary if all ~~of~~ it's statistical properties does not change with time. Then these ~~are~~ called as stationary random process otherwise non-stationary random process.

statistical Independence:

two processes $x(t)$ & $Y(t)$ are statistically independent if the random variable group $x(t_1), x(t_2), \ldots x(t_n)$ is independent ~~of~~ the group $Y(t_1'), Y(t_2') \ldots Y(t_n')$

for any choice of $t_1, t_2, \ldots t_N$ and $t_1', t_2', \ldots t_M'$ independence requires that the joint density be factorable by groups.

i.e., $f_{XY}(x_1, x_2, \ldots x_N, y_1, y_2, \ldots y_M; t_1, t_2 \ldots t_N, t_1', t_2' \ldots t_M')$

$\Rightarrow f_X(x_1, x_2 \ldots x_N; t_1, t_2 \ldots t_N)\, f_Y(y_1, y_2 \ldots y_M; t_1', t_2' \ldots t_M')$

Distribution & Density functions:-

The distribution & density function for a random process $X(t)$ is given as

for a particular value of time t_1 the distribution is associated with a random variable $x_1 = x(t_1)$ will be denoted as $F_X(x_1, t_1)$

i.e., $F_X(x_1, t_1) = P\{x(t_1) \leq x_1\}$, for any real number x_1

for two random variables

$x_1 = x(t_1)$ & $x_2 = x(t_2)$ then the joint distribution can be given as

$F_X(x_1, x_2)$ $F_X(x_1, x_2; t_1, t_2) = P\{x(t_1) \leq x_1, x(t_2) \leq x_2\}$

Illy for N no. of random variables $x_i = x(t_i)$ then the joint distribution function can be given as

$F_X(x_1, x_2 \cdots x_n, t_1 t_2 \cdots t_n) = P\{X(t_1) \le x_1, X(t_2) \le x_2$
$\cdots X(t_n) \le x_n\}$

form the above information the joint density function can be given as

$$f_X(x_1 t_1) = \frac{d}{dx} F_X(x_1, t_1)$$

$$f_X(x_1 x_2; t_1 t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_X(x_1 x_2; t_1, t_2)$$

$$f_X(x_1, x_2 \cdots x_n; t_1, t_2 \cdots t_n) = \frac{\partial^n}{\partial x_1 \partial x_2 \cdots \partial x_n} F_X(x_1, x_2 \cdots x_n; t_1, t_2 \cdots t_n)$$

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First order stationary process:-

A Random process is called stationary of order one if it's first order density function does not change with time.

i.e., $\Delta = t_2 - t_1$ change in time

$$\therefore f_X(x_1, t_1) = f_X(x_1, t_1 + \Delta) \quad \text{——} ①$$

eq ① must be true for any t_1 and any real number Δ, if $x(t)$ is to be first order stationary process

In the above equation $x(x_1, t_1)$ is independent

~~of~~ t_1 and process ~~of~~ mean value $\cancel{E[x(t)]=\bar{x}}$ =

~~cons~~ $E[x(t)] = \bar{x} = \text{constant}$ —— ②

for a random variable x_1 and x_2, the mean values are given as

WKT $x(t_1) = \cancel{X_1}$

$x(t_2) = \cancel{X_2}$

$\therefore E[x_1] = E$

$E[x(t_1)] = E[x_1] = \overline{x_1}$

$E[x(t_2)] = E[x_2] = \overline{x_2}$

$$E[x(t_1)] = \int_{-\infty}^{\infty} x_1 f_x(x_1, t_1)\, dx_1 \text{ —— ③}$$

$$E[x(t_2)] = \int_{-\infty}^{\infty} x_2 f_x(x_2, t_2)\, dx_2 \text{ —— ④}$$

by letting $\Delta = t_2 - t_1 \Rightarrow t_2 = t_1 + \Delta$

$\therefore E[x(t_2)] = E[x(t_1+\Delta)]$

from eq ① $E[x(t_2)] = E[x(t_1)]$

$\therefore E[x(t_1)] = E[x(t_1+\Delta)]$ which must be a constant because t_1 & Δ are arbitary

Second order stationary process:

A random process is called stationary to order two if it's second order density function does not change with time

i.e., $f_X(x_1,t_1; x_2,t_2) = f_X(x_1,t_1+\Delta; x_2,t_2+\Delta)$

for all values of t_1, t_2, Δ

Now the correlation $E[X_1 X_2] = E[X(t_1).X(t_2)]$ of a random process will be a function of t_1 & t_2 and it can be replace with Auto correlation function of a random process $x(t)$.

$\therefore \quad R_{XX}(\tau) = E[X(t_1).X(t_2)]$ let $\tau = t_2 - t_1$

$t_2 = t_1 + \tau$

$$\boxed{R_{XX}(\tau) = E[X(t_1)X(t_1+\tau)]}$$

34 **wide sence stationary process:- (WSS):-**

A random process is said to be wide sence stationary process if it satisfies the following two conditions.

i.e., 1) $E[X] = \bar{X}$ = constant

2) $R_{XX}(\tau) = E[X(t_1)X(t_1+\tau)]$

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strict sence stationary process (SSS):-

The extension random process concept to 'N' no. of random variables. i.e., $X_i = X(t_i)$ where $i = 1, 2, 3, \ldots N$, can be given as a random process of stationarity N then it's N^{th} order density function is invarient to the time origin.

$$f_x(x_1, x_2, \cdots x_n; t_1, t_2 \cdots t_n) = f_x(x_1, x_2 \cdots x_n; t_1+\Delta, t_2+\Delta + \cdots t_n+\Delta)$$

for all values of $t_1, t_2 \cdots t_n$ and Δ

A random process stationary to all orders, $N = 1, 2, \cdots$ is called strict sence stationary

Time averages & Erogodicity :-

The time average for a quantity is defined as follows

$$A[x(t)] = \bar{x}$$

i.e., $$\bar{x} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt$$

where A is called time average function & $\bar{x}$ is called mean of the time average function.

The Auto correlation function can be given as

$$R_{xx}(\tau) = A[x(t)\, x(t+\tau)]$$

i.e., $$R_{xx}(\tau) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t+\tau)\, d\tau$$

Erogodic Random process:

An erogodic random process is a process that satisfies the following two conditions.

i. $A[x(t)] = E[x(t)]$

i.e., $\bar{x} = \overline{X}$

where $\bar{x}$ is the time average value of a function $x(t)$ and $\overline{X}$ is called mean

ii. $R_{XX}(\tau) = R_{xx}(\tau)$

i.e., $E[x(t) \cdot x(t+\tau)] = A[x(t) \cdot x(t+\tau)]$

where $\overline{X}$ & $R_{XX}(\tau)$ are statistical averages &

$\bar{x}$ & $R_{xx}(\tau)$ are time averages.

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Mean Erogodic Random process:

If the statistical average $\overline{X}$ of a random process $X(t)$ equals to the time average of any sample function $x(t)$ with probability '1' for all sample functions then the process $X(t)$ is called mean erogodic random process.

$$A[x(t)] = E[X(t)]$$

i.e., $\bar{x} = \overline{X}$

to satisfy the above relation it should

meet four condition.

1. $x(t)$ has a infinite constant mean $\bar{x}$ for all values of $x(t)$

2. $x(t)$ is bounded

i.e. $\int |x(t)\, dt| < \infty$ for all values of t

3. $$\underset{T \to \infty}{Lt} \frac{1}{2T} \int_{-T}^{T} x(t)\, dt < \infty$$

4. $E[x^2(t)] = R_{xx}(t_1)(t_1, t_2) < \infty$

correlation Erogodic Random process:-

Let $x(t)$ be a stationary continuous process with auto correlation function $R_{xx}(\tau)$ then it is called correlation erogodic process if and only if it satisfys the following condition for all values of τ

i.e., $R_{xx}(\tau) = A[x(t) \cdot x(t+\tau)]$

$$= \underset{T \to \infty}{Lt} \Big]$$

$$= \underset{T \to \infty}{Lt} \frac{1}{2T} \int_{-T}^{T} x(t)\, x(t+\tau)\, d\tau$$

let $\omega(t) = x(t) \cdot x(t+\tau)$ then

$$E[\omega(t)] = E[x(t) \cdot x(t+\tau)] =$$
$$= R_{xx}(\tau)$$

lly $R_{\omega\omega}(\tau) = E[\omega(t) \cdot \omega(t+\tau)]$

$$= E\left[x(t)\cdot x(t+\tau)\cdot x(t+\tau)\cdot x(t+\tau+\tau)\right]$$

$$= E\left[x(t)\cdot x(t+\tau)\cdot x(t)\cdot x(k+\tau)\right]$$

where K is time constant from the above equation the ~~ortho~~ auto correlation erogodicity requires forth order moments of $x(t)$.

the condition for WSS sequence $x(n)$ to be auto correlation erogodic for all values of n.

$$R_{xx}(N) = \underset{N\to\infty}{Lt} \frac{1}{2N+1}\int_{-n}^{n} x(n)\, x(n+N)\, dN$$

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Auto correlation:-

The Auto correlation process is the correlation of two random variables $x_1 = x(t_1)$, $x_2 = x(t_2)$, both are define at times t_1 & t_2 respectively then $R_{xx}(t_1, t_2) = E[x(t_1)x(t_2)]$

$$\Rightarrow R_{xx}(\tau) = E[x(t)\cdot x(t+\tau)]$$

Properties of Auto correlation functions:-

i, The mean square value of a random process can be obtain from $R_{xx}(\tau)$ by substituting $\tau = 0$

Proof :- w.k.t $R_{xx}(\tau) = E[x(t)\,x(t+\tau)]$

$$R_{xx}(0) = E[x(t)x(t+0)]$$

$$= E[x(t)\cdot x(t)]$$

$$= E[x^2(t)]$$

ii, $R_{xx}(\tau) = R_{xx}(-\tau)$

Proof :-

w.k.T

$$R_{xx}(\tau) = E[x(t)\,x(t+\tau)]$$

$$R_{xx}(-\tau) = E[x(t)\cdot x(t-\tau)] \quad \left[\begin{array}{l}\text{let } t-\tau=u \\ t=u+\tau\end{array}\right]$$

$$= E[x(u+\tau)\cdot x(u)]$$

$$= E[x(u)\,x(u+\tau)]$$

$$= R_{xx}(\tau)$$

$$\therefore R_{xx}(-\tau) = R_{xx}(\tau)$$

iii, $R_{xx}(\tau)$ is bounded by its value about origin

i.e. $|R_{xx}(\tau)| \le R_{xx}(0)$

Proof :- let $E\left|[x(t_1)+x(t_2)]^2\right| \ge 0$

$$E[x^2(t_1)] + E[x^2(t_2)] + 2E[x(t_1)x(t_2)] \ge 0$$

$$E[x^2(t_1)] + E[x^2(t_1)] + 2R_{xx}(\tau) \geq 0$$

$$2|E[x^2(t_1)]| + 2|R_{xx}(\tau)| \geq 0$$

$\tau = 0$

$t_2 - t_1 = 0$

$\boxed{t_1 = t_2}$

$$2|R_{xx}(\tau)| \geq 2|E[x^2(t_1)]|$$

$$R_{xx}(\tau) \geq E[x^2(t_1)]$$

$$R_{xx}(0) \geq R_{xx}(\tau) \text{ [from property (1)]}$$

iv, If a process $z(t) = x(t) + Y(t)$ where $x(t)$ & $Y(t)$ are random processe then $R_{zz}(\tau) = R_{xx}(\tau) + R_{YY}(\tau) + R_{xY}(\tau) + R_{Yx}(\tau)$

Proof :- we know that

$$R_{zz}(\tau) = E[z(t) \cdot z(t+\tau)]$$

$$z(t) = E[z(t) \cdot z(t+\tau)]$$
$$x(t) + Y(t)$$

$$z(t+\tau) = E[x(t+\tau) + Y(t+\tau)]$$

$$R_{zz}(\tau) = E[[x(t) + Y(t)][x(t+\tau) + Y(t+\tau)]]$$

$$= E[x(t)x(t+\tau) + x(t)Y(t+\tau) + Y(t)x(t+\tau) + Y(t) \cdot Y(t+\tau)]$$

$$= E[x(t) \cdot x(t+\tau)] + E[x(t) \cdot Y(t+\tau)] + E[Y(t) \cdot x(t+\tau)] + E[Y(t) \cdot Y(t+\tau)]$$

$$R_{ZZ}(\tau) = R_{XX}(\tau) + R_{XY}(\tau) + R_{YX}(\tau) + R_{YY}(\tau)$$

v) If $x(t)$ is a periodic then it's Auto correlation function is also periodic.

Proof:- $R_{XX}(\tau) = E[x(t) \cdot x(t+\tau)]$

$$R_{XX}(\tau+\tau_0) = E[x(t) \cdot x(t+\tau+\tau_0)]$$

$$= E[x(t) \cdot x(t+\tau)]$$

$$= R_{XX}(\tau)$$

$$R_{XX}(\tau) = R_{XX}(\tau+\tau_0)$$

vi) If a random process has no periodic components and $x(t)$ is non zero mean then $R_{XX}(\tau) = E[\bar{x}^2]$

Proof:-

we have $R_{XX}(\tau) = E[x(t_1) \cdot x(t_2)]$

where $x(t_1) = x_1$

$x(t_2) = x_2$

If there are no periodic components as $|\tau| \longrightarrow \infty$ & x_1 & x_2 can be considered as independent random variables

$$\therefore R_{XX}(\tau) = E[x_1 x_2]$$

$\because$ x_1 & x_2 are from same random processes at two time instances then $E[x_1] = E[x_2] = E[x]$

$$\therefore R_{xx}(\tau) = E[x \cdot x] \quad E[x] \cdot E[x]$$

$$= E[x^2]$$

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cross correlation function:

The cross correlation function of two random process $x(t)$ & $Y(t)$ can be given as

$$R_{xy}(t_1, t_2) = E[x(t_1)\ Y(t_2)]$$

$$= E[x(t_1)\ Y(t_1+\tau)]$$

$$= E[x(t) \cdot Y(t+\tau)]$$

$$= R_{xy}(\tau)$$

properties of cross correlation function:

i, $R_{xy}(\tau) = R_{yx}(-\tau)$

Proof:- $R_{yx}(\tau) = E[y(t)\ x(t+\tau)]$

$$R_{yx}(-\tau) = E[y(t)\ x(t-\tau)]$$

$t - \tau = u$

$t = u + \tau$

$$= E[y(u+\tau)\ x(u)]$$

$$= E[x(u)\ y(u+\tau)]$$

$$= R_{xy}(\tau)$$

$$\therefore R_{xy}(\tau) = R_{yx}(-\tau)$$

(iii) If $x(t)$ & $y(t)$ are two random process and $R_{xx}(\tau)$ & $R_{yy}(\tau)$ are their respective Auto correlation function then $|R_{xy}(\tau)| \leq \sqrt{R_{xx}(0) R_{yy}(0)}$

Proof :-

let $$E\left[\frac{x(t)}{\sqrt{R_{xx}(0)}} + \frac{y(t)}{\sqrt{R_{yy}(0)}}\right]^2 \geq 0$$

$$\frac{E[x(t)]^2}{R_{xx}(0)} + \frac{E[y(t)]^2}{R_{yy}(0)} + \frac{2E[x(t)y(t)]}{\sqrt{R_{xx}(0) R_{yy}(0)}} \geq 0$$

$$\frac{R_{xx}(0)}{R_{xx}(0)} + \frac{R_{yy}(0)}{R_{yy}(0)} + \frac{2R_{xy}(\tau)}{\sqrt{R_{xx}(0) R_{yy}(0)}} \geq 0$$

$$1 + 1 + \frac{2R_{xy}(\tau)}{\sqrt{R_{xx}(0) R_{yy}(0)}} \geq 0$$

$$2 + \frac{2R_{xy}(\tau)}{\sqrt{R_{xx}(0) R_{yy}(0)}} \geq 0$$

$$2\left[1 + \frac{R_{xy}(\tau)}{\sqrt{R_{xx}(0) R_{yy}(0)}}\right] \geq 0$$

$$1 + \frac{R_{xy}(\tau)}{\sqrt{R_{xx}(0) R_{yy}(0)}} \geq 0$$

$$\frac{\sqrt{R_{xx}(0)R_{yy}(0)} + R_{xy}(\tau)}{\sqrt{R_{xx}(0)R_{yy}(0)}} \geq 0$$

$$\left|\sqrt{R_{xx}(0)R_{yy}(0)}\right| + |R_{xy}(\tau)| \geq 0$$

$$\sqrt{R_{xx}(0)R_{yy}(0)} \geq |R_{xy}(\tau)| \geq 0$$

iii, If two random processes $x(t)$ & $y(t)$ are independent then $R_{xy}(\tau) = E[x(t)] \cdot E[y(t)]$

Proof :- we know that

$$R_{xy}(\tau) = E[x(t)\,y(t+\tau)]$$

If two random variables are independent then

$E[x\,y] = E[x] \cdot E[y]$

$$\Rightarrow R_{xy}(\tau) = E[x(t)]\,E[y(t+\tau)]$$

If $y(t)$ is periodic then $y(t) = y(t+\tau)$

$$\Rightarrow R_{xy}(\tau) = E[x(t)]\,E[y(t)]$$

iv, If two random processes $x(t)$ and $y(t)$ are zero mean then limit $\tau \to \infty$ $R_{xy}(\tau)$

$$R_{xy}(\tau) = \underset{\tau \to \infty}{Lt} R_{yx}(\tau) = 0$$

Proof:- we know that

$$R_{XY}(\tau) = E[x(t)\,Y(t+\tau)]$$

$$R_{YX}(\tau) = E[Y(t)\,x(t+\tau)]$$

If $\tau \to \infty$ $x(t)$ & $Y(t)$ are independent events

$$\lim_{\tau\to\infty} R_{XY}(\tau)$$

$$\lim_{\tau \to \infty} R_{XY}(\tau) = E[x(t)]\,E[Y(t+\tau)]$$

$$= E[x(t)] \cdot E[Y(t)]$$

bcz $x(t)$ & $Y(t)$ are 0 means then

$$\lim_{\tau\to\infty} R_{XY}(\tau) = 0 \cdot 0$$

$$\Rightarrow 0$$

$$\lim_{\tau\to\infty} R_{YX}(\tau)$$

$$\lim_{\tau\to\infty} R_{YX}(\tau) = E[Y(t)]\,E[x(t+\tau)]$$

$$= E[Y(t)] \cdot E[x(t+\tau)]$$

as $x(t)$ & $Y(t)$ are zero means than

$$\lim_{\tau\to\infty} R_{YX}(\tau) = 0 \cdot 0$$

$$\Rightarrow 0$$

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Gaussian Random process

consider a continuous random process for N no. of random variables, $X_1 = x(t_1)$, $X_2 = x(t_2) \ldots$ $X_N = x(t_n)$ corresponds to the time instances of $t_1, t_2 \ldots t_n$

i.e., $N = 1, 2, 3 \ldots$ any time instant $t_1, t_2, \ldots t_N$

these random variables are jointly Gaussian then their joint Gaussian density function can be given as

i.e., $f_X(x_1, x_2 \ldots x_N; t_1, t_2 \ldots t_N) = \dfrac{1}{\sqrt{CX(2\pi)^n}} \exp\left\{ -\frac{1}{2} \ldots \right.$

$$= \frac{1}{\sqrt{CX(2\pi)^n}} \exp\left\{ \frac{-1}{2} (X-\bar{X})(X-\bar{X})^T (CX^{-1}) \right\}$$

where $[X - \bar{X}] = \begin{bmatrix} X_1 - \bar{X}_1 \\ X_2 - \bar{X}_2 \\ \vdots \\ X_N - \bar{X}_N \end{bmatrix}$ and

$$[CX] = \begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdots & C_{1N} \\ C_{21} & C_{22} & C_{23} & \cdots & C_{2N} \\ \vdots & & & & \\ C_{N1} & C_{N2} & C_{N3} & \cdots & C_{NN} \end{bmatrix}$$

The mean value of $x(t_i)$ is

$$\bar{X}_i = E[X(t_i)] = \bar{X}(t_i)$$

The elements of covarience matrix $[CX]$ can be given as

$$C_{XX} = E[(X-\bar{X})(X-\bar{X})]$$

$$C_{X_i X_j} = E[(X_i - \bar{X}_i)(X_j - \bar{X}_j)]$$

$$= E[(X_i - E(X_i))(X_j - E(X_j))]$$

$$= E[X(t_i) - E[X(t_i)])(X(t_j) - E[X(t_j)]]$$

$$\Rightarrow E[X(t_i) \cdot X(t_j)] \Rightarrow$$

$E[x(t_i)]\, E[x(t_j)]$

$\Rightarrow C_{xx}(t_i, t_j)$, which is called Auto co-varience function.

By extending the above equation the Auto co-varience function can also be written as

$$C_{xx} = R_{xx} - \bar{x}\cdot\bar{x}$$

$$\Rightarrow C_{xx}(t_i, t_j) = R_{xx}(t_i, t_j) - E[x(t_i)]\, E[x(t_j)]$$

If the Gaussian random process is not a stationary, mean, Auto co-varience function depends on absolute time

for a wide sence stationary the mean value must be a constant

i.e., $\bar{x}_i = \bar{x}(t_i) \Rightarrow E[x(t_i)]$

$$\Rightarrow \bar{x}$$

while the Auto covarience function depends on only time ~~variences~~ differences and not on absolute time i.e., $C_{xx}(t_i, t_j) = C_{xx}(t_i - t_j)$

$$R_{xx}(t_i, t_j) = R_{xx}(t_i - t_j)$$

Poisson Random process:-

Poission Random process is an example for discrete random process

* It discribes the no. of times that som event has occured as a function of time, where the events occured at random times

Ex:- emission of electrons from the surface of a light sensetive material (photo detecter)

Note:- The poission Random process is also called as poission counting process.

* To define poisson process we shall require two conditions.

1, the event occur only once in any vanising time of interval

2, The occurance times must be statastically independent so that the number that occurs in any given time interval is independent of number in any other non over lapping time interval.

Poission distribution & density functions:

the poission random variable x has it's density and distribution function hey

$$f_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} \delta(x-k) \quad \text{--- (1)}$$

$$F_X(x) = e^{-b} \sum_{k=0}^{\infty} \frac{b^k}{k!} u(x-k) \quad \text{--- (1)}$$

where $b>0$ is a real constant and these function appear quite similar to Binomial Random variable

If $N=\infty$ and $p=0$

where $= \lambda T$

11/09/17 5. Random process spectral characteristics

Power density spectrum & it's properties.

The spectral properties of a determistic signal $x(t)$ are containing in it's fourier transform $X(\omega)$ and is given by

$$X(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

where the function $X(\omega)$ is some times called as the spectrum of $x(t)$ and has the units of volts/Hz if $x(t)$ is a voltage signal.

i.e., considered as voltage density spectrum

If $X(\omega)$ is well known then $x(t)$ can be recovered by means of Inverse fourier transform and can be given as

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega$$

If in place of voltage we use power of a random process then it is commonly known as power density spectrum.

power density spectrum

for a random process $x(t)$, let $x_T(t)$ can be define as a portion of sample function. $x(t)$. That exist for the interval $[-T, T]$

i.e., $x_T(t) = \begin{cases} x(t) & -T \le t \le T \\ 0 & \text{else} \end{cases}$ —— ①

As long as T is finite we represent $x_T(t)$ has bounded variation and can be given as

$$\int_{-\infty}^{\infty} |x(t)|dt < \infty$$

$$\int_{-T}^{T} |x_T(t)|dt < \infty \quad \text{—— ②}$$

The Fourier transform of $x_T(t)$ can be given as

$$x(\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$x(\omega) = \int_{-T}^{T} x_T(t) e^{-j\omega t} dt \quad \text{—— ③}$$

The energy content in $x(t)$ over the interval $[-T, T]$

can be given as $E = \int_{-\infty}^{\infty} |x(t)|^2 dt$

$$\Rightarrow \int_{-T}^{T} |x_T(t)|^2 dt \quad \text{—— ④}$$

using the parsevals theorem the above Eq can also be re-written as

$$E(t) = \int_{-T}^{T} |x_T(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |x_T(\omega)|^2 d\omega \quad \text{--- (5)}$$

by dividing the above Eq by $2T$ we obtain average power over the Interval $[-T, T]$

$$\underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-\infty}^{\infty} |x(t)|^2 dt = P(t)$$

$$\Rightarrow \underset{T\to\infty}{Lt} \cdot \frac{\frac{1}{2\pi}\int_{-\infty}^{\infty} |x_T(\omega)|^2 d\omega}{2T}$$

In the above equation $|x_T(\omega)|^2$ is called power density spectrum.

* As long as $T\to\infty$ and take the expected value to obtain a suitable power density spectrum

$$P(t) = \underset{T\to\infty}{Lt} \int_{-T}^{T} \frac{E[x^2(t)]}{2T} dt$$

$$P_{xx}(\omega) = \underset{T\to\infty}{Lt} \frac{1}{2\pi}\int_{-T}^{T} \frac{E[X_T(\omega)]^2}{2T} d\omega$$

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sate state and prove the properties of Auto density spectrum.

Note : [$S_{XX}(\omega) = \lim_{T \to \infty} \frac{1}{2T} E|X_T(\omega)|^2$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j\omega\tau}\, d\tau$$

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{j\omega\tau}\, d\omega \;]$$

Property.1:

. for a WSS power spectral density (PSD) at zero frequency gives the area under the graph of a Auto correlation function.

Proof:- $R_{XX}(\tau) \xrightarrow{F\cdot T} S_{XX}(\omega)$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j\omega\tau}\, d\tau ,$$

$$S_{XX}(\omega)\Big|_{\omega=0} = \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-j(0)\tau}\, d\tau$$

$$= \int_{-\infty}^{\infty} R_{XX}(\tau)\, d\tau$$

i.e., It is equal to the area under the graph of auto correlation function.

2) The mean square value of WSS random process is equal to the area under the graph of PSD

Proof: WKT

$$R_{XX}(\tau) \xleftrightarrow{F.T} S_{XX}(\omega)$$

$$S_{XX}(\omega) = \int_{-\infty}^{\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau \quad \text{--- (1)}$$

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega \quad \text{--- (2)}$$

WKT

$$R_{XX}(\tau) = E[x(t) \cdot x(t+\tau)]$$

$$= E[x(t) \cdot x(t+0)]$$

$$= E[x^2(t)] \quad \text{--- (3)}$$

from Eq (2)

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega$$

$$R_{XX}(\tau)\Big|_{\tau=0} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega(0)} d\omega$$

$$R_{XX}(\tau)\Big|_{\tau=0} \Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) d\omega$$

from Eq (3)

$$E[x^2(t)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) d\omega$$

3) The power spectrum density of derivative of random process is ω^2 times the power spectral density.

Proof:- WKT

$$S_{XX}(\omega) = \underset{T\to\infty}{Lt} \frac{1}{2T} |X_T(\omega)|^2 d\omega \quad \text{--- (1)}$$

$$X_T(\omega) = \int_{-T}^{T} x_T(t) e^{-j\omega t} dt$$

$$= \int_{-T}^{T} \frac{d}{dt} x_T(t) e^{-j\omega t} dt$$

$$\Rightarrow \int_{-T}^{T} x_T(t) \frac{d}{dt} \cdot e^{-j\omega t} dt$$

$$X_T(\omega) = \int_{-T}^{T} x_T(t) e^{-j\omega t} (-j\omega) dt$$

$$= (-j\omega) \int_{-T}^{T} x_T(t) e^{-j\omega t} dt$$

$$\left.\begin{aligned} X_T(\omega) &= (-j\omega) X_T(\omega) \\ X_T(-\omega) &= (j\omega) X_T(-\omega) \end{aligned}\right\} \text{(2)}$$

(2) in (1)

$$S_{XX}(\omega) = \underset{T\to\infty}{Lt} \frac{1}{2T} |X_T(\omega)|^2 d\omega$$

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} |X_T(\omega) X_T(-\omega)| d\omega$$

from Eq (2)

$$= \underset{T\to\infty}{Lt} \frac{1}{2T} |(-j\omega) X_T(\omega) (j\omega) X_T(-\omega)| d\omega$$

$$\Rightarrow \underset{T\to\infty}{Lt} \frac{1}{2T} \omega^2 |X_T(\omega) X_T(-\omega)| d\omega$$

$$\rightarrow \omega^2 \underset{T\to\infty}{Lt} \frac{1}{2T} |X_T(\omega)|^2 d\omega$$

13/9/17 $\boxed{S_{\dot{X}\dot{X}}(\omega) = \omega^2 S_{XX}(\omega)}$

Relation b/w power spectrum and auto correlation function (or) wiener khintchine relation.

statement: The Inverse transform of power spectral density is the time average of autocorrelation function. i.e.,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega) e^{j\omega t} d\omega = A[R_{XX}(t, t+\tau)] \quad \text{—} (0)$$

If $x(t)$ is constant continuous

$$X_T(t) = \begin{cases} x(t) & -T \le t \le T \\ 0 & \text{else} \end{cases} \quad \text{—} (1)$$

If 't' finite, $x(t)$ is bounded

$$\int_{-\infty}^{\infty} |x(t)| dt < \infty \Rightarrow \int_{-T}^{T} |x_T(t)| dt < \infty$$

F.T of $x(t)$

$$X(\omega) = FT[x(t)] = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt$$

$$X_T(\omega) = \int_{-T}^{T} x_T(t)\, e^{-j\omega t_1}\, dt_1 \quad \text{---} (2)$$

wkT

$$S_{XX}(\omega) = \underset{T\to\infty}{Lt} \frac{1}{2T} E\left|X_T(\omega)\right|^2 d\omega \quad \text{---} (3)$$

Eq (3) can be re written as

$$S_{XX}(\omega) = \underset{T\to\infty}{Lt} \frac{1}{2T} E\left| X_T(\omega)\, X_T(-\omega)\right| d\omega \quad \text{---} (4)$$

from eq (2)

$$X_T(-\omega) = \int_{-T}^{T} x_T(t)\, e^{-j(-\omega)t_2} dt_2$$

$$= \int_{-T}^{T} x_T(t_2)\, e^{j\omega t_2}\, dt_2 \quad \text{---} (5)$$

(2) & (5) in eq (4)

$$S_{XX}(\omega) = \underset{T\to\infty}{Lt} \frac{1}{2T} E\left| \int_{-T}^{T} x_T(t_1) e^{-j\omega t_1} dt_1 \int_{-T}^{T} x_T(t_2) e^{j\omega t_2} dt_2 \right| d\omega$$

$$\Rightarrow \underset{T\to\infty}{Lt} \frac{1}{2T} E\left[\int_{-T}^{T}\int_{-T}^{T} x_T(t_1)\, x_T(t_2)\, e^{-j\omega t_1} e^{-j\omega t_2}\, dt_1\, dt_2 \right] d\omega$$

$$\Rightarrow \underset{T\to\infty}{Lt} \frac{1}{2T} E\left[\int_{-T}^{T}\int_{-T}^{T} x_T(t_1)\, x_T(t_2)\, e^{-j\omega(t_1 - t_2)}\, dt_1\, dt_2 \right] d\omega$$

$$S_{XX}(\omega) = \underset{T\to\infty}{Lt} \frac{1}{2T} E\left[\int_{-T}^{T} x_T(t_1)\, x_T(t_2)\, e^{-j\omega\tau}\, d\tau \right] d\omega \quad \text{---} (6)$$

eq ⓐ in eq ⓑ

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)e^{j\omega t}\,d\omega$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} E\left|\underset{T\to\infty}{Lt}\frac{1}{2T}\int_{-T}^{T} x_T(t_1)x_T(t_2)e^{-j\omega\tau}\,d\tau\right)d\omega e^{j\omega t}\,d\omega$$

$$\Rightarrow \underset{T\to\infty}{Lt}\frac{1}{2T}\frac{1}{2\pi}\int_{-\infty}^{\infty} E\left[\int_{-T}^{T} x_T(t_1)x_T(t_2)e^{-j\omega\tau}e^{j\omega t}\,d\tau\,d\omega\right.$$

$$\Rightarrow \underset{T\to\infty}{Lt}\frac{1}{2T}E\left[\int_{-T}^{T} x_T(t_1)x_T(t_2)\,dt\right]\int_{-\infty}^{\infty}\frac{1}{2\pi}e^{-j\omega(\tau-t)}\,d\omega$$

$$\Rightarrow \underset{T\to\infty}{Lt}\frac{1}{2T}E\left[\int_{-T}^{T} x_T(t_1)x_T(t_2)\,d\tau\right.$$

$$\Rightarrow \underset{T\to\infty}{Lt}\frac{1}{2T}\int_{-T}^{T} E[x(t_1)x(t_2)]\,dt_1\,dt_2$$

$$\Rightarrow \underset{T\to\infty}{Lt}\frac{1}{2T}\int_{-T}^{T} R_{xx}(t_1,t_2)\,d\tau)$$

$$= A[R_{xx}(t_1,t_2)]$$

$$\therefore \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{xx}(\omega)e^{j\omega t}\,d\omega = A[R_{xx}(t,t+\tau)]$$

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Properties of cross power density spectrum

Some of the properties of cross pds of real random process $x(t)$ & $y(t)$ are listed below.

1, $S_{XY}(\omega) = S_{YX}(-\omega) \Rightarrow S^*_{YX}(\omega)$

2, Real function of $S_{XY}(\omega)$

$Re[S_{XY}(\omega)]$ is a even function of ω

3, $Im[S_{XY}(\omega)]$ & $Im[S_{YX}(\omega)]$ are odd functions of ω

4, $S_{XY}(\omega)=0$ & $S_{YX}(\omega)=0$ when $x(t)$ and $y(t)$ are orthogonal functional

5, If $x(t)$ and $y(t)$ are uncorrelated and have constant means $\bar{x}$ and $\bar{y}$ then $S_{XY}(\omega) = S_{YX}(\omega) \Rightarrow 2\pi(\bar{x}\,\bar{y})$

6, $A[R_{XY}(t, t+\tau)] \xleftrightarrow{F.T} S_{XY}(\omega)$

$A[R_{YX}(t, t+\tau)] \xleftrightarrow{F.T} S_{YX}(\omega)$

7, $S_{XY}(\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau) \cdot e^{-j\omega t} \, d\tau$

$S_{YX}(\omega) = \int_{-\infty}^{\infty} R_{YX}(\tau) \, e^{-j\omega t} \, d\tau$

$R_{XY}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{XY}(\omega) \, e^{j\omega t} \, d\omega$

$R_{YX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_{YX}(\omega) \, e^{j\omega t} \, d\omega$

Relation b/w cross power density spectrum and cross correlation function.

statement: The time average cross correlation function of two random variables x and y and power density spectrum are fourier transform pairs

$$S_{XY}(\omega) = \int_{-\infty}^{\infty} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_{XY}(\tau)\, e^{-j\omega\tau}\, d\tau \quad \text{--- (1)}$$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega) e^{j\omega t}\, d\omega = A\left[R_{XY}(t, t+\tau)\right] \quad \text{--- (2)}$$

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{1}{2T} E\left[x_T^*(\omega)\, y_T(\omega)\right] d\omega \quad \text{--- (3)}$$

$$x_T(t) = \begin{cases} x(t) & -T<t<T \\ 0 & \text{else} \end{cases} \qquad y_T(t) = \begin{cases} y(t) & -T<t<T \\ 0 & \text{else} \end{cases}$$

$$\left.\begin{aligned} x_T^*(\omega) &= x_T(-\omega) = \int_{-T}^{T} x_T(t_1) e^{j\omega t_1} dt_1 \\ y_T(\omega) &= \int_{-T}^{T} y_T(t_2) e^{-j\omega t_2} dt_2 \end{aligned}\right\} \text{--- (4)}$$

eq (4) in eq (3)

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{1}{2T} E\left[\int_{-T}^{T} x_T(t_1) e^{j\omega t_1} dt_1 \int_{-T}^{T} y_T(t_2) e^{-j\omega t_2} dt_2\right] d\omega$$

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T}\int_{-T}^{T} E\left[x_T(t_1) y_T(t_2)\right] e^{j\omega t_1} e^{-j\omega t_2} dt_1\, dt_2$$

$$\Rightarrow \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XY}(\tau)\, e^{-j\omega(t_2-t_1)}\, dt_1\, dt_2$$

$$S_{XY}(\omega) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_{XY}(\tau)\, e^{-j\omega\tau}\, d\tau \quad \text{—— (5)}$$

from eq (2)

L.H.S

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XY}(\omega)\, e^{j\omega t}\, d\omega \Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty} \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} R_{XY}(\tau)\, e^{-j\omega\tau}\, d\tau\, e^{j\omega t}\, d\omega$$

$$\Rightarrow \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_{XY}(\tau)\, d\tau\, \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega\tau}\, e^{j\omega t}\, d\omega$$

$$\Rightarrow \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_{XY}(\tau)\, d\tau\, \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega(\tau-t)}\, d\omega$$

$$\Rightarrow \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} R_{XY}(\tau)\, d\tau\, (1)$$

$$\boxed{\because \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-j\omega(\tau-t)}\, d\omega = 1}$$

$$\Rightarrow A\left[R_{XY}(\tau)\right]$$

$$A\left[R_{XY}(t, t+\tau)\right] \Rightarrow \text{RHS}$$

UNIT - VI

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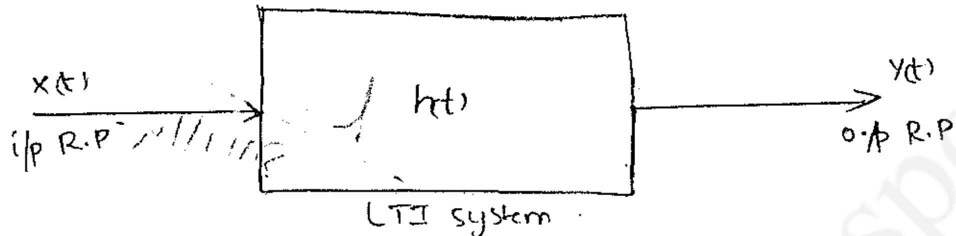
* UNIT-5 *

* Response of Linear Systems To Random Signals *

→ **Output Response of Linear System or Response of Linear Time Invariant system (LTI):**

Let a random process $X(t)$ is applied to an LTI system whose impulse response is $h(t)$, as shown in figure



$Y(t)$ be the output random process. Then the o/p response of the linear system is defined by

$$\boxed{\text{o/p response of LTI system} = Y(t) = h(t) \circledast X(t) = \int_{-\infty}^{\infty} h(\tau)\, X(t-\tau)\, d\tau}$$

or

$$Y(t) = X(t) \circledast h(t) = \int_{-\infty}^{\infty} X(\tau)\, h(t-\tau)\, d\tau$$

Here $X(t)$ is i/p random process, $h(t)$ is impulse response of LTI system and $Y(t)$ is o/p random process.

→ **Mean value of o/p response:**

The o/p response of LTI system is defined by

$$Y(t) = h(t) * X(t) = \int_{-\infty}^{\infty} h(\tau)\, X(t-\tau)\, d\tau$$

Apply mean (expectation) operator, we get,

$$E[Y(t)] = E[h(t) * X(t)]$$

$$= E\left[\int_{-\infty}^{\infty} h(\tau)\, X(t-\tau)\, d\tau\right]$$

$$= \int_{-\infty}^{\infty} h(\tau)\, E[x(t-\tau)]\, d\tau$$

$\because x(t)$ is a WSS R.P. then $E[x(t)] = E[x(t+\tau)]$

$$= \bar{x} = \text{constant}.$$

$$E[Y(t)] = \int_{-\infty}^{\infty} h(\tau)\, \bar{x} \cdot d\tau$$

$$E[Y(t)] = \bar{x} \int_{-\infty}^{\infty} h(\tau) \cdot d\tau$$

We know frequency response of LTI system $= H(\omega)$

$$= F[h(\tau)] = \int_{-\infty}^{\infty} h(\tau)\, e^{-j\omega\tau}\, d\tau$$

Zero frequency response of LTI system $= H(0) = \int_{-\infty}^{\infty} h(\tau)\, e^{-j(0)\tau} d\tau$

$$\Rightarrow H(0) = \int_{-\infty}^{\infty} h(\tau)\, d\tau$$

So now, $E[Y(t)] = \bar{x} \int_{-\infty}^{\infty} h(\tau)\, d(\tau)$

$$\bar{Y} = \bar{x} \cdot H(0)$$

$$\boxed{\therefore \bar{Y} = E[Y(t)] = H(0) \cdot \bar{x} = \text{The mean value of o/p respon}}$$

Thus, the mean value of o/p response is equal to th product of the mean value of i/p random process and zero frequency-response of the LTI system.

$\Rightarrow$ Mean square value of o/p response:

The o/p response of LTI system is defined by

$$Y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau_1)\, x(t-\tau_1)\, d\tau = \int_{-\infty}^{\infty} h(\tau_2)\, x(t-\tau_2)\, d$$

The meansquare value of o/p response $= E[Y^2(t)]$

$$= E[Y(t) \cdot Y(t)]$$

$$= E[(h(t) * x(t))(h(t) * x(t))]$$

$$= E\left[\left(\int_{-\infty}^{\infty} h(\tau_1)\, x(t-\tau_1)\, d\tau_1\right)\left(\int_{-\infty}^{\infty} h(\tau_2)\, x(t-\tau_2)\, d\tau_2\right)\right]$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t-\tau_1)\, x(t-\tau_2)\, h(\tau_1)\, h(\tau_2)\, d\tau_1\, d\tau_2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\left[x(t-\tau_1)\, x(t-\tau_2)\right] h(\tau_1)\, h(\tau_2)\, d\tau_1 d\tau_2$$

We know that

$$E[x(t-\tau_1)\, x(t-\tau_2)] = R_{xx}\left[(\cancel{t}-\tau_2)-(\cancel{t}-\tau_1)\right] = R_{xx}(\tau_1-\tau_2)$$

$\because$ For WSS R.P, $R_{xx}(t_1, t_2) = E[x(t_1)x(t_2)] = R_{xx}(t_2-t_1)$

$$\therefore \boxed{E[Y^2(t)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{xx}(\tau_1-\tau_2)\, h(\tau_1)\, h(\tau_2)\, d\tau_1\, d\tau_2}$$

= The meansquare value of o/p response.

The meansquare value of o/p response is a function of 'τ' only, not absolute time t.

*Types of Random Processes:

There are four types of random processes.

(a) Low pass random processes

(b) Band pass random processes.

(c) Band limited random processes

(d) Narrow Band random processes.

(a) Low Pass Random Processes:

A random process is defined as the low pass random process $X(t)$, if its power spectral density $S_{xx}(\omega)$ has significant components

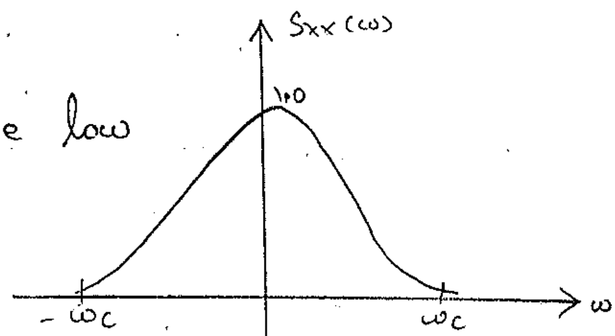


Fig: Power Spectrum of LPRP

within the frequency band as shown in figure. for example base band signals, such as speech, image and video are low pass random processes.

(b) Band Pass Random Processes :

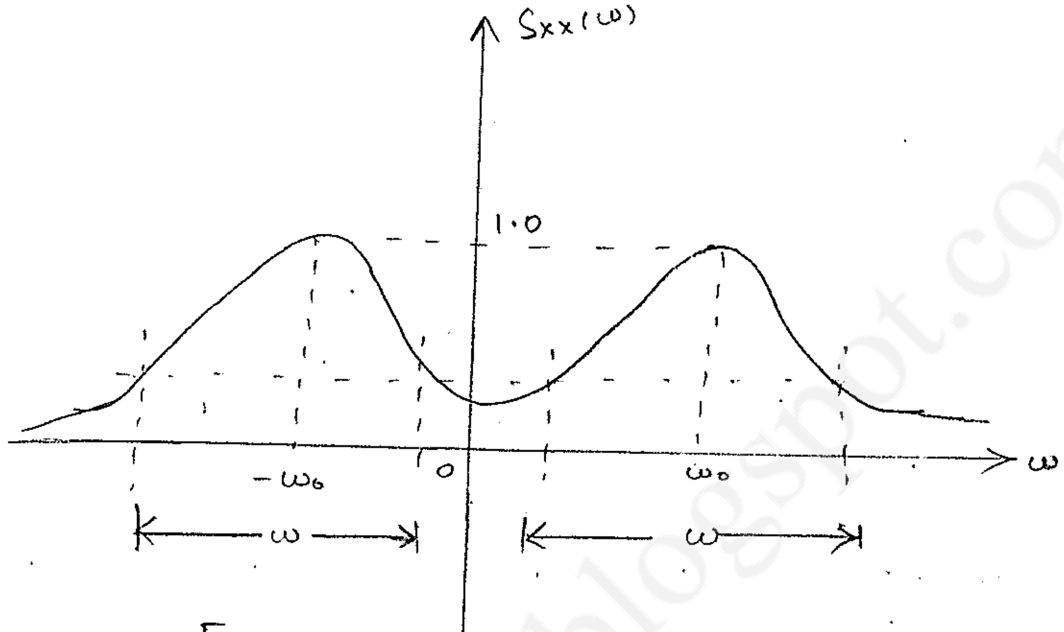


Fig: Power spectrum of BPRP

A random process X(t) is called a band pass random process if its power spectral density $S_{XX}(\omega)$ has its significant component within a bandwith W that does not include $\omega=0$, (as shown in figure). But in practise, the spectrum may have a small amount of power spectrum at $\omega=0$, as shown in figure. The spectral components outside the band W are very small and can be neglected. For example, modulated signals with carrier frequency ω_0 and bandwidth W are band pass random processes.

The noise transmitting over a communication channel can be modelled as a band pass random process.

(c) Band Limited Random Processes:

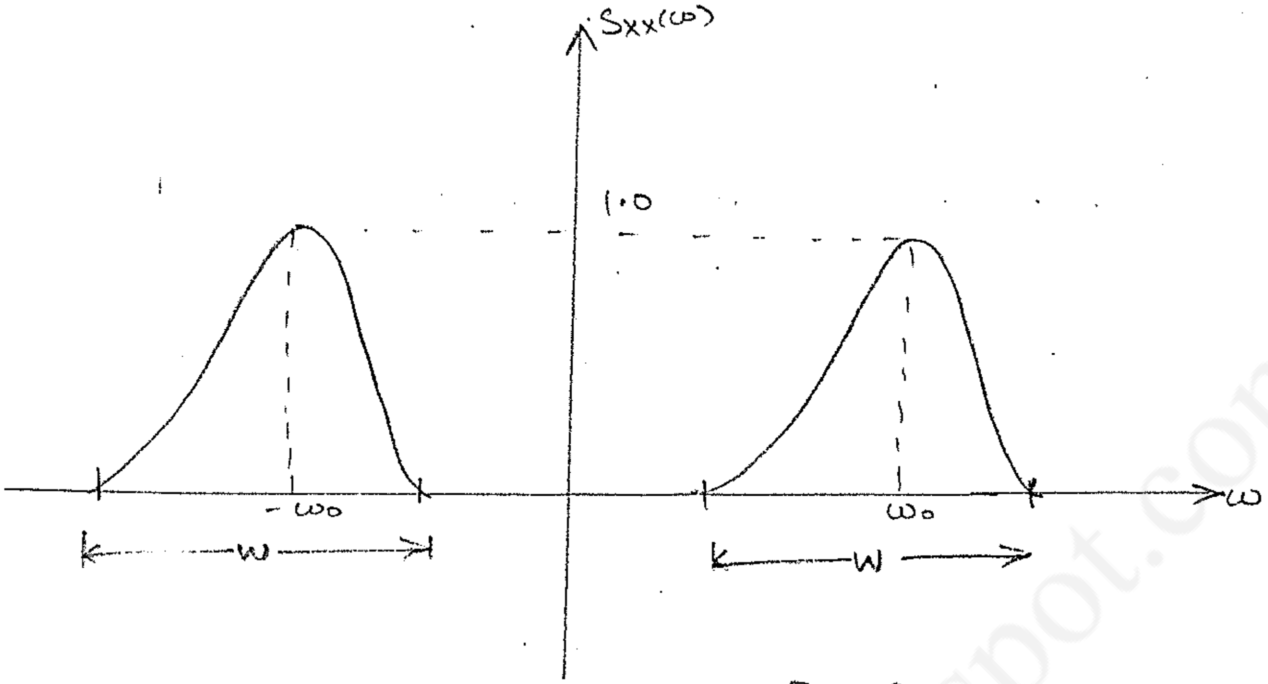


Fig: Power spectrum of BLRP

A band pass random process is said to be band limited random process if its power spectrum components are zero outside the frequency band of width W that doesnot include $\omega = 0$. The power spectrum density of Band limited random process is as shown in fig.

(d) Narrow Band Random Processes:

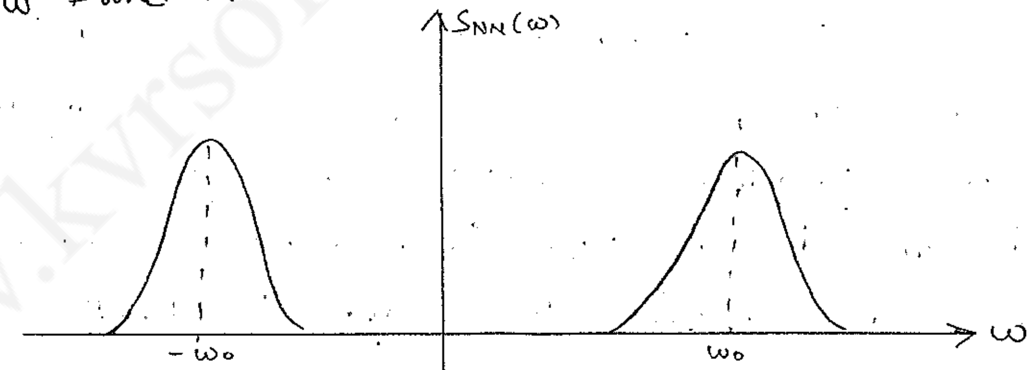
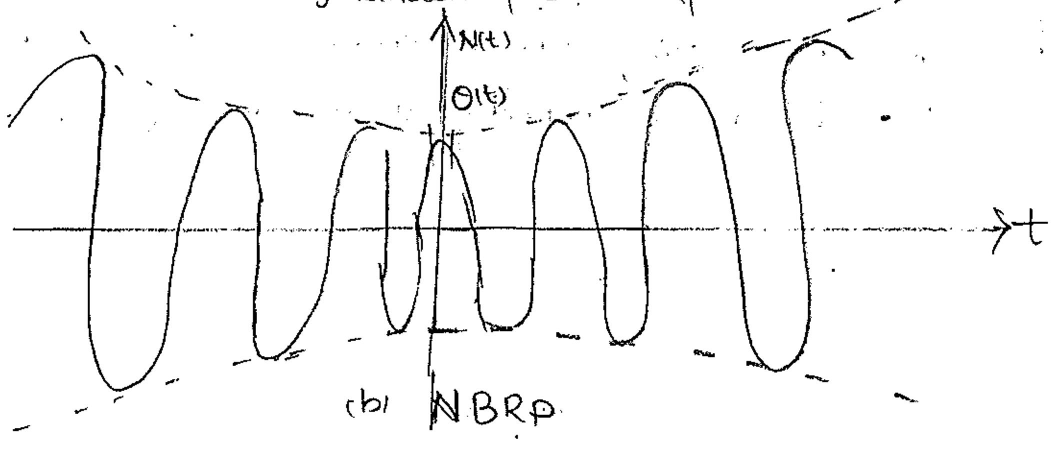


Fig: (a) Power spectrum of NBRP



(b) NBRP

A band limited random process is said to be a narrow band process if the bandwidth W is very small, compared to the band central frequency, i.e. $W << W_0$ where W = bandwidth and w_0 is the frequency at which the power spectrum is maximum.

The power density of a narrow band random process $N(t)$ is as shown in fig(a).

The narrow band process can be modelled as a cosine function slowly varying in amplitude and phase with frequency w_0 as shown in fig(b). It can be expressed as

$$N(t) = A(t) \cos[w_0 t + \theta(t)]$$

where $A(t)$ - amplitude of R.P.

$\theta(t)$ - phase of R.P.

* Representation of Narrow Band Random Process:

For any arbitrary WSS random process $N(t)$

$$N(t) = A(t) \cos[w_0 t + \theta(t)]$$

$$= A(t) [\cos w_0 t \cos\theta(t) - \sin w_0 t \sin\theta(t)]$$

$$N(t) = A(t) \cos(\theta(t)) \cos(w_0 t) - A(t) \sin\theta(t) \sin w_0 t \rightarrow ①$$

The Quadrature form of Narrowband process is defined by

$$N(t) = X(t) \cos(w_0 t) - Y(t) \sin(w_0 t) \longrightarrow ②$$

Comparing ① and ②, we have

The inphase component of $N(t) = X(t) = A(t) \cos[\theta(t)]$

The Quadrature phase component of $N(t) = Y(t) = A(t) \sin[\theta(t)]$

The relationship b/n the processes $A(t)$ and $\theta(t)$ are given by

$$X^2(t) + Y^2(t) = A^2(t)\cos^2[\theta(t)] + A^2(t)\sin^2[\theta(t)]$$

$$X^2(t) + Y^2(t) = A^2(t)(\cos^2[\theta(t)] + \sin^2[\theta(t)])$$

$$= A^2(t)\cdot 1$$

$$\Rightarrow X^2(t) + Y^2(t) = A^2(t)$$

$$\boxed{A(t) = \sqrt{X^2(t) + Y^2(t)}}$$

$$\frac{Y(t)}{X(t)} = \frac{A(t)\sin[\theta(t)]}{A(t)\cos[\theta(t)]}$$

$$\frac{Y(t)}{X(t)} = \tan\theta[(t)]$$

$$\Rightarrow \boxed{\theta(t) = \tan^{-1}\left[\frac{Y(t)}{X(t)}\right]}$$

Imp * **Properties of Band limited Random Process:**

* Let $N(t)$ be any band limited WSS R.P. with zero mean value and power spectrum density $S_{NN}(\omega)$. If the random process is represented by $N(t) = X(t)\cos\omega_0 t - Y(t)\sin\omega_0 t$ then some important properties of BLRP are given below.

1. If $N(t)$ is WSS; then $X(t)$ and $Y(t)$ are jointly widesense stationary rp's

2. If $N(t)$ has zero mean i.e $E[N(t)] = 0$ then.
$$E[X(t)] = E[Y(t)] = 0$$

3. The mean square values of the processes are equal i.e, $E[N^2(t)] = E[X^2(t)] = E[Y^2(t)]$

4. Both processes $X(t)$ and $Y(t)$ have same autocorrelation functions i.e., $R_{XX}(\tau) = R_{YY}(\tau)$

5. The cross-correlation function of $X(t)$ and $Y(t)$ are given by $R_{XY}(\tau) = -R_{YX}(\tau)$.

   If the processes are orthogonal then

$$R_{XY}(\tau) = R_{YX}(\tau) = 0.$$

6. Both $X(t)$ and $Y(t)$ have same power spectral densities.

$$S_{YY}(\omega) = S_{XX}(\omega) = \begin{cases} S_N(\omega-\omega_0) + S_N(\omega+\omega_0) & ; |\omega| \leq W_0 \\ 0 & ; \text{otherwise} \end{cases}$$

7. The cross-power spectrums of r.p's $S_{XY}(\omega) = -S_{YX}(\omega)$.

8. If $N(t)$ is a Gaussian random process then $X(t)$ and $Y(t)$ are jointly Gaussian. r.

9. The relationship b/n ACF and PSD $[S_{NN}(\omega)]$ is

$$R_{XX}(\tau) = \frac{1}{\pi}\int_0^{\infty} S_{NN}(\omega)\cos[(\omega-\omega_0)\tau]\,d\omega$$

$$R_{XY}(\tau) = \frac{1}{\pi}\int_0^{\infty} S_{NN}(\omega)\sin[(\omega-\omega_0)\tau]\,d\omega$$

10. If $N(t)$ is zero mean Gaussian and its PSD, $S_{NN}(\omega)$ is symmetric about $\pm\omega_0$, then $X(t)$ and $Y(t)$ are statistically independent.

* Cross Correlation Function of O/P Response:

(i) The CCF b/n $X(t)$ and $Y(t)$ = $R_{xy}(\tau) = E[X(t)\,Y(t+\tau)]$

O/p response of linear system $Y(t) = h(t) * x(t)$

$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Put $\tau = \tau_1$, we get

$$= \int_{-\infty}^{\infty} h(\tau_1)\, x(t-\tau_1)\, d\tau_1$$

$$Y(t+\tau) = \int_{-\infty}^{\infty} h(\tau_1)\, x(t+\tau-\tau_1)\, d\tau_1$$

$$R_{xy}(\tau) = E\left[x(t) \int_{-\infty}^{\infty} h(\tau_1)\, x(t+\tau-\tau_1)\, d\tau_1\right]$$

$$= \cancel{E}\left[\int_{-\infty}^{\infty} h(\tau_1)\, E[x(t)\, x(t+\tau-\tau_1)]\right] d\tau_1$$

We know ~~that~~ If $x(t)$ is WSS then

$$R_{xx}(t_1, t_2) = E[x(t_1)\, x(t_2)] = R_{xx}(t_2 - t_1)$$

$$E[x(t) \cdot x(t+\tau-\tau_1)] = R_{xx}((t+\tau-\tau_1) - t)$$

$$= R_{xx}(\tau - \tau_1)$$

Now $R_{xy}(\tau) = \int_{-\infty}^{\infty} h(\tau_1)\, R_{xx}(\tau-\tau_1)\, d\tau_1$

$$\boxed{\therefore R_{xy}(\tau) = h(\tau) * R_{xx}(\tau)}$$

(ii) The CCF b/n $Y(t)$ and $X(t)$ = $R_{yx}(\tau) = E[Y(t)\, x(t+\tau)]$

O/p response of LTI $Y(t) = h(t) * x(t)$

$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

Put $\tau = \tau_1$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, x(t-\tau_1)\, d\tau_1$$

$$R_{YX}(\tau) = E\left[\int_{-\infty}^{\infty} h(\tau_1)\, x(t-\tau_1)\, d\tau_1 \cdot x(t+\tau)\right]$$

$$= E\left[\int_{-\infty}^{\infty} h(\tau_1)\, x(t-\tau_1)\, x(t+\tau)\, d\tau_1\right]$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, E\left[x(t-\tau_1)\, x(t+\tau)\right] d\tau_1$$

We know that if $x(t)$ is WSS then

$$R_{xx}(t_1, t_2) = E[x(t_1)\cdot x(t_2)] = R_{xx}(t_2 - t_1)$$

$$R_{xx}(t-\tau_1, t+\tau) = E[x(t-\tau_1)x(t+\tau)] = R_{xx}[(t+\tau) - (t-\tau_1)]$$

$$= R_{xx}(\tau + \tau_1)$$

Now $$R_{xy}(\tau) = \int_{-\infty}^{\infty} h(\tau_1)\, R_{xx}(\tau+\tau_1)\, d\tau_1$$

Put $\tau_1 = -\tau_2 \Rightarrow d\tau_1 = -d\tau_2$

$$= \int_{\infty}^{-\infty} h(-\tau_2)\, R_{xx}(\tau + (-\tau_2)) - d\tau_2$$

$$= -\int_{-\infty}^{+\infty} h(-\tau_2)\, R_{xx}(\tau_1 - \tau_2) - d\tau_2$$

$$= \int_{-\infty}^{\infty} h(-\tau_2)\, R_{xx}(\tau_1 - \tau_2)\, d\tau_2$$

$$\boxed{\therefore R_{yx}(\tau) = h(-\tau) * R_{xx}(\tau)}$$

(iii) The ACF of $y(t) = R_{yy}(\tau) = E[y(t)\, y(t+\tau)]$.

o/p response of LTI $y(t) = h(t) * x(t)$

$$= \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, x(t-\tau_1)\, d\tau_1$$

$$R_{yy}(\tau) = E\left[\int_{-\infty}^{\infty} h(\tau_1)\, x(t-\tau_1)\, d\tau_1 \quad y(t+\tau)\right]$$

$$= E\left[\int_{-\infty}^{\infty} h(\tau_1)\; x(t-\tau_1)\; y(t+\tau)\, d\tau_1\right]$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\; E\left[x(t-\tau_1)\, y(t+\tau)\right] d\tau_1$$

We know if $x(t)$ & $y(t)$ are jointly WSS then

$$R_{xy}(t_1, t_2) = E\left[x(t_1)\, y(t_2)\right] = R_{xy}(t_2 - t_1)$$

$$E\left[x(t-\tau_1)\; y(t+\tau)\right] = R_{xy}\left((t+\tau) - (t-\tau_1)\right) = R_{xy}(\tau + \tau_1)$$

$$\Rightarrow R_{yy}(\tau) = \int_{-\infty}^{\infty} h(\tau_1)\; R_{xy}(\tau + \tau_1)$$

Replace τ_1 by $-\tau_2$ $\Rightarrow$ $d\tau_1 = -d\tau_2$

$$= \int_{+\infty}^{-\infty} h(-\tau_2)\; R_{xy}(\tau + (-\tau_2))\, (-d\tau_2)$$

$$= -\int_{-\infty}^{\infty} h(-\tau_2)\; R_{xy}(\tau - \tau_2)\; (-d\tau_2)$$

$$= \int_{-\infty}^{\infty} h(-\tau_2)\; R_{xy}(\tau - \tau_2)\; d\tau_2$$

$$\therefore \boxed{R_{yy}(\tau) = h(-\tau) * R_{xy}(\tau)}$$

(v) The ACF of $Y(t) = R_{YY}(\tau) = E[Y(t)\, Y(t+\tau)]$

o/p response of linear system $= Y(t) = h(t) * X(t)$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, x(t-\tau)\, d\tau$$

$$\Rightarrow Y(t+\tau) = \int_{-\infty}^{\infty} h(\tau_1)\, x((t+\tau)-\tau_1)\, d\tau_1$$

$$R_{YY}(\tau) = E\left[Y(t) \int_{-\infty}^{\infty} h(\tau_1)\, x((t+\tau)-\tau_1)\, d\tau_1 \right]$$

$$= E\left[\int_{-\infty}^{\infty} h(\tau_1)\, Y(t)\, x((t+\tau)-\tau_1)\, d\tau_1 \right]$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, E[Y(t)\, X(t+\tau-\tau_1)]\, d\tau_1$$

We know, If $x(t)$ and $Y(t)$ are jointly WSS then

$$R_{YX}(t_1, t_2) = E[Y(t_1)\, X(t_2)] = R_{YX}(t_2 - t_1)$$

$$E[Y(t)\, X(t+\tau-\tau_1)] = R_{YX}((t+\tau-\tau_1)-t) = R_{YX}(\tau-\tau_1)$$

$$\Rightarrow R_{YY}(\tau) = \int_{-\infty}^{\infty} h(\tau_1)\, R_{YX}(\tau-\tau_1)\, d\tau_1$$

$$\boxed{\therefore R_{YY}(\tau) = h(\tau) * R_{YX}(\tau) = R_{YX}(\tau) * h(\tau)}$$

*Auto Correlation Function Response of Linear System:

The autocorrelation function o/p response of system is defined by

$$ACF \text{ of } Y(t) = R_{YY}(t_1,t_2) = E[Y(t_1)\ Y(t_2)].$$

O/p response of linear system $Y(t) = h(t) * x(t)$

$$= \int h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\alpha)\, x(t-\alpha)\, d\alpha$$

$$Y(t_1) = h(t_1) * X(t_1) = \int_{-\infty}^{\infty} h(\tau)\, x(t_1-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau_1)\, x(t_1-\tau_1)\, d\tau_1$$

$$Y(t_2) = h(t_2) * X(t_2) = \int_{-\infty}^{\infty} h(\tau)\, x(t_2-\tau)\, d\tau = \int_{-\infty}^{\infty} h(\tau_2)\, x(t_2-\tau_2)\, d\tau_2$$

$$R_{YY}(t_1,t_2) = E\left[\int_{-\infty}^{\infty} h(\tau_1)\, X(t_1-\tau_1)\, d\tau_1 \cdot \int_{-\infty}^{\infty} h(\tau_2)\, X(t_2-\tau_2)\, d\tau_2\right]$$

$$= E\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\, h(\tau_2)\, x(t_1-\tau_1)\, X(t_2-\tau_2)\, d\tau_1\, d\tau_2\right]$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)\, h(\tau_2)\, E\left[x(t_1-\tau_1)\ x(t_2-\tau_2)\right] d\tau_1\, d\tau_2$$

We know that if $x(t)$ is WSS random process then

$$R_{xx}(t_1,t_2) = E[x(t_1)\, X(t_2)] = R_{xx}(t_2-t_1)$$

$$E[x(t_1-\tau_1)\, x(t_2-\tau_2)] = R_{xx}((t_2-\tau_2)-(t_1-\tau_1))$$

$$= R_{xx}((t_2-t_1)+\tau_1-\tau_2)$$

$$= R_{xx}((t_2-t_1)+\tau_1-\tau_2)$$

$$\Rightarrow R_{YY}(t_1,t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)R_{XX}(t_2-t_1+\tau_1-\tau_2)\,d\tau_1 d\tau_2$$

For WSS random process $t_2-t_1=\tau$

$$R_{YY}(t_1,t_2) = R_{YY}(t_2-t_1) = R_{YY}(\tau)$$

$$\boxed{\therefore R_{YY}(t_1,t_2) = R_{YY}(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2)\,h(\tau_1)h(\tau_2)\,d\tau_1 d\tau_2}$$

$$\boxed{R_{YY}(\tau) = h(-\tau) * R_{XX}(\tau) * h(\tau) = R_{XX}(\tau)*h(\tau)*h(-\tau)}$$

From CCF properties (i) and (iii)

$$R_{XY}(\tau) = h(\tau) * R_{XX}(\tau)$$

$$R_{YY}(\tau) = h(-\tau) * R_{XY}(\tau)$$

then $R_{YY}(\tau) = h(-\tau) * \left(h(\tau) * R_{XX}(\tau)\right) = h(-\tau) * h(\tau) * R_{XX}(\tau)$

$$R_{YY}(\tau) = R_{XX}(\tau) * h(\tau) * h(-\tau)$$

$\therefore$ ACF of o/p response is a function of τ only not absolute time 't'. Hence, ACF of o/p response is also a WSS random process.

*** Power Spectrum Density of o/p response of LTI system,**

(OR)

Test

Derive relationship b/n PSD of input and output random process of linear system or LTI system.

Statement: If the PSD of input random process $X(t)$ is $S_{XX}(\omega)$ and $H(\omega)$ is transfer function of linear system, then the PSD of o/p response of linear system is given by

$$\boxed{S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)}$$ i.e.,

o/p PSD of $Y(t)$ = $|H(\omega)|^2$ i/p PSD of $X(t)$

Proof: The ACF of o/p response of LTI system is defined by $y(t) = R_{YY}(\tau) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} R_{XX}(\tau+\tau_1-\tau_2)\, h(\tau_1) h(\tau_2)\, d\tau_1\, d\tau_2$

Apply fourier transform on both sides, we get

$$F[R_{YY}(\tau)] = F\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2) h(\tau_1) h(\tau_2)\, d\tau_1\, d\tau_2\right]$$

$$= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2)\, h(\tau_1)\, h(\tau_2)\, d\tau_1\, d\tau_2\right] e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau_1) \int_{-\infty}^{\infty} h(\tau_2) \left[\int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2)\, e^{-j\omega\tau} d\tau\right] d\tau_2\, d\tau_1$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, e^{j\omega\tau_1} \int_{-\infty}^{\infty} h(\tau_2)\, e^{-j\omega\tau_2} \int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2)\, e^{-j\omega\tau} e^{-j\omega\tau_1} e^{j\omega\tau_2}\, d\tau\, d\tau_2\, d\tau_1$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, e^{j\omega\tau_1} \int_{-\infty}^{\infty} h(\tau_2)\, e^{-j\omega\tau_2} \left[\int_{-\infty}^{\infty} R_{XX}(\tau+\tau_1-\tau_2)\, e^{-j\omega(\tau+\tau_1-\tau_2)}\, d\tau\right] d\tau_2\, d\tau_1$$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, e^{j\omega\tau_1} \int_{-\infty}^{\infty} h(\tau_2)\, e^{-j\omega\tau_2} \left[\int_{-\infty}^{\infty} R_{XX}(t)\, e^{-j\omega t}\, dt\right] d\tau_2\, d\tau_1$$

Put $\tau+\tau_1-\tau_2 = t$

$d\tau = dt$

$\tau \to -\infty \Rightarrow t = -\infty$

$\tau \to \infty \Rightarrow t = \infty$

$$= \int_{-\infty}^{\infty} h(\tau_1)\, e^{j\omega\tau_1} \int_{-\infty}^{\infty} h(\tau_2)\, e^{-j\omega\tau_2}\, F[R_{XX}(t)]\, d\tau_2\, d\tau_1$$

We know $R_{XX}(t) \xleftrightarrow{F.T} S_{XX}(\omega)$

$$R_{YY}(\tau) \longleftrightarrow S_{YY}(\omega)$$

Now, $$S_{YY}(\omega) = \int_{-\infty}^{\infty} h(\tau_1) e^{j\omega\tau_1} \int_{-\infty}^{\infty} h(\tau_2) e^{-j\omega\tau_2} S_{XX}(\omega)\, d\tau_2\, d\tau_1$$

$\because$ independent operator

$$= S_{XX}(\omega) \left(\int_{-\infty}^{\infty} h(\tau_1) e^{j\omega\tau_1} d\tau_1\right)\left(\int_{-\infty}^{\infty} h(\tau_2) e^{-j\omega\tau_2} d\tau_2\right)$$

We know that $F[h(\tau)] = H(\omega) = \int_{-\infty}^{\infty} h(\tau) e^{-j\omega\tau} d\tau$

Replacing $\tau = \tau_2$

$$F[h(\tau_2)] = H(\omega) = \int_{-\infty}^{\infty} h(\tau_2) e^{-j\omega\tau_2} d\tau_2$$

Replacing $\tau = \tau_1$

$$F[h(\tau_1)] = H(\omega) = \int_{-\infty}^{\infty} h(\tau_1) e^{-j\omega\tau_1} d\tau_1$$

$$H^*(\omega) = \int_{-\infty}^{\infty} h^*(\tau_1) \left(e^{-j\omega\tau_1}\right)^* d\tau_1$$

$\because$ Conjugation of real function is real.

$$H^*(\omega) = \int_{-\infty}^{\infty} h(\tau_1) e^{j\omega\tau_1} d\tau_1$$

Now, we have

$$S_{YY}(\omega) = S_{XX}(\omega)\; H^*(\omega)\; H(\omega)$$

$$= S_{XX}(\omega)\, |H(\omega)|^2 \qquad \left\{\because x^*(t)\, x(t) = |x(t)|^2\right.$$

$$\therefore S_{YY}(\omega) = |H(\omega)|^2\, S_{XX}(\omega)$$

Hence proved.

* Cross Power Spectral Densities:

(i) $S_{XY}(\omega) = H(\omega) \cdot S_{XX}(\omega)$

Proof: We know $R_{XY}(\tau) = h(\tau) * R_{XX}(\tau)$ {From case (i) in CCF o/p response}

Applying fourier transform, we have

$$F[R_{XY}(\tau)] = F[h(\tau) * R_{XX}(\tau)]$$

We know that, $F[g_1(t) * g_2(\tau)] = F[g_1(\tau)] \cdot F[g_2(\tau)]$
$= G_1(\omega) \cdot G_2(\omega)$

$$h(\tau) \xleftrightarrow{F.T} H(\omega)$$

$$R_{XX}(\tau) \longleftrightarrow S_{XX}(\omega)$$

$$R_{XY}(\tau) \longleftrightarrow S_{XY}(\omega)$$

$$\Rightarrow F[R_{XY}(\tau)] = F[h(\tau)] \cdot F[R_{XX}(\tau)]$$

$$\boxed{\therefore S_{XY}(\omega) = H(\omega) \cdot S_{XX}(\omega)}$$

Hence proved.

(ii) $S_{YX}(\omega) = H^*(\omega) S_{XX}(\omega) = H(-\omega) S_{XX}(\omega)$

Proof: We know $R_{YX}(\tau) = h(-\tau) * R_{XX}(\tau)$ {from case (ii) in CCF o/p response}

Applying fourier transform, we have

$$F[R_{YX}(\tau)] = F[h(-\tau) * R_{YX}(\tau)]$$

We know that $F[g_1(t) * g_2(t)] = F[g_1(t)] \cdot F[g_2(t)]$
$= G_1(\omega) \cdot G_2(\omega)$

$$h(\tau) \xleftrightarrow{F.T} H(\omega) \Rightarrow h(-\tau) \longleftrightarrow H(-\omega) = H^*(\omega)$$

$$R_{YX}(\tau) \longleftrightarrow S_{YX}(\omega)$$

$$R_{XX}(\tau) \longleftrightarrow S_{XX}(\omega)$$

$$\Rightarrow F[R_{YX}(\tau)] = F[h(-\tau)] \cdot F[R_{XX}(\tau)]$$

$$\boxed{\therefore S_{YX}(\omega) = H^*(\omega)\, S_{XX}(\omega)}$$

Hence proved.

II method:

$$S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

Proof. We know $R_{YY}(\tau) = h(-\tau) * h(\tau) * R_{XX}(\tau)$

{ From ACF o/p respon

Apply fourier transform on both sides, we get

$$F[R_{YY}(\tau)] = F[h(-\tau) * [h(\tau) * R_{XX}(\tau)]]$$

We have $F[g_1(\tau) * g_2(\tau)] = F[g_1(\tau)] \cdot F[g_2(\tau)]$

$$= G_1(\omega) \cdot G_2(\omega)$$

$$F[R_{YY}(\tau)] = F[h(-\tau)] \cdot F[h(\tau) * R_{XX}(\tau)]$$

Again by the property, we have

$$F[h(\tau) * R_{XX}(\tau)] = F[h(\tau)] \cdot F[R_{XX}(\tau)]$$

$$\Rightarrow F[R_{YY}(\tau)] = F[h(-\tau)] \cdot F[h(\tau)] \cdot F[R_{XX}(\tau)]$$

$$h(-\tau) \longleftrightarrow H^*(\omega)$$

$$h(\tau) \longleftrightarrow H(\omega)$$

$$R_{XX}(\tau) \longleftrightarrow S_{XX}(\omega)$$

$$R_{YY}(\tau) \longleftrightarrow S_{YY}(\omega)$$

$$\Rightarrow S_{YY}(\omega) = H^*(\omega)\, H(\omega)\, S_{XX}(\omega)$$

$$\therefore S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

where $S_{XX}(\omega) \Rightarrow$ PSD of i/p R.P. $X(t)$

$H(\omega) \Rightarrow$ Transfer function or frequency response of LTI system

$S_{YY}(\omega) \Rightarrow$ PSD of o/p R.P $Y(t)$

Hence proved.

*Problems:

1. Let a random process $X(t)$ having PSD $S_{XX}(\omega) = \frac{3}{49+\omega^2}$ is applied to a network whose impulse response is $h(t) = t^2 \exp(-7t)u(t)$ and o/p response is represented by $Y(t)$.

(i) Find average power of i/p random process $X(t)$.

(ii) Find PSD of o/p random process $Y(t)$.

(iii) Find average power of o/p random process $Y(t)$.

Sol: Given $S_{XX}(\omega) = \frac{3}{49+\omega^2}$

Impulse response of the given n/w $= h(t) = t^2 \exp(-7t)\, u(t)$

(i) Average power of i/p random process $X(t) = P_{XX}$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(\frac{3}{49+\omega^2}\right) d\omega$$

$$= \frac{1}{2\pi}\cdot 3\int_{-\infty}^{\infty} \left(\frac{1}{49+\omega^2}\right) d\omega$$

$$= \frac{1}{2\pi}\cdot 3\int_{-\infty}^{\infty} \left(\frac{1}{\omega^2+7^2}\right) d\omega$$

$$= \frac{3}{2\pi}\left[\frac{1}{7}\tan^{-1}\left(\frac{\omega}{7}\right)\right]_{-\infty}^{\infty} \quad \because \int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$= \frac{3}{14\pi}\left[\tan^{-1}\left(\frac{\omega}{7}\right)\right]_{-\infty}^{\infty}$$

$$= \frac{3}{14\pi}\left[\tan^{-1}\left(\frac{\infty}{7}\right) - \tan^{-1}\left(\frac{-\infty}{7}\right)\right]$$

$$= \frac{3}{14\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{3}{14\pi}\left[\frac{2\pi}{2}\right]$$

$$\boxed{\therefore P_{XX} = \frac{3}{14}\, W = 0.214 \text{ Watts}}$$

(ii) PSD of o/p random process $Y(t)$:

We know relationship b/w input, output PSDs of random process i.e,

$$\text{Output PSD of } Y(t) = S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$$

The given impulse response $h(t) = t^2 \cdot e^{-7t} u(t)$

We know $t^n e^{-at} u(t) \xleftrightarrow{F.T} \dfrac{n!}{(a+j\omega)^{n+1}}$

Here $n=2$, $a=7$ then $t^2 e^{-7t} u(t) \longleftrightarrow \dfrac{2!}{(7+j\omega)^{2+1}}$

$$= \frac{2}{(7+j\omega)^3}$$

and $h(t) \longleftrightarrow H(\omega)$

$$F[h(t)] = H(\omega) = F[t^2 e^{-7t} u(t)] = \frac{2}{(7+j\omega)^3}$$

$$|H(\omega)| = \left|\frac{2}{(7+j\omega)^3}\right|$$

$$= \frac{2}{|(7+j\omega)|^3}$$

$$= \frac{2}{\sqrt{(7^2+\omega^2)^3}}$$

$$|H(\omega)| = \frac{2}{(7^2+\omega^2)^{3/2}} \Rightarrow |H(\omega)|^2 = \left(\frac{2}{(7^2+\omega^2)^{3/2}}\right)^2$$

$$= \frac{4}{(7^2+\omega^2)^3}$$

$$\therefore |H(\omega)|^2 = \frac{4}{(49+\omega^2)^3}$$

Now $S_{YY}(\omega) = \left[\dfrac{4}{(49+\omega^2)^3}\right]\left[S_{XX}(\omega)\right]$

$$= \left[\frac{4}{(49+\omega^2)^3}\right]\left[\frac{3}{(49+\omega^2)}\right]$$

$$= \frac{12}{(49+\omega^2)^4}$$

$$\boxed{\therefore S_{YY}(\omega) = \frac{12}{(7^2+\omega^2)^4}}$$

(iii) Average power of o/p random process $Y(t)$ is

$$P_{YY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{12}{(7^2+\omega^2)^4}\,d\omega$$

$$= \frac{6}{\pi}\int_{-\infty}^{\infty} \frac{1}{(7^2+\omega^2)^4}\,d\omega \qquad \because \int \frac{1}{(a^2+\omega^2)^4}\,dx = \frac{5\pi}{16a^7}$$

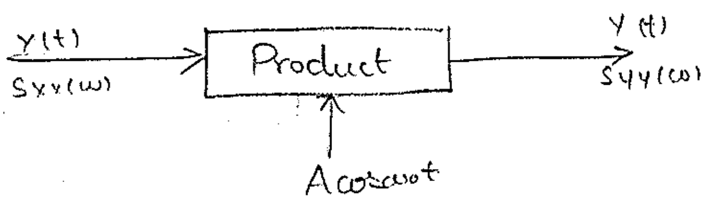
$$= \frac{\cancel{6}^{\,3}}{\cancel{\pi}} \cdot \frac{5\cancel{\pi}}{\cancel{16}_{\,8}\times(7)^7}$$

$$= \frac{15}{8\times(7)^7}$$

$$= 2.27\times10^{-6}\ \text{W}$$

$$\boxed{\therefore P_{YY} = 2.27\ \mu\text{W}}$$

2. Let $X(t)$ is input random process, find $R_{YY}(\tau)$ and $S_{YY}(\omega)$ in terms of i/p PSD $S_{XX}(\omega)$ for the product device s/w shown in figure. ~~Find~~



Sol: Output random process $= Y(t) =$ Output of product device

$$= X(t)\,A\cos\omega_0 t = A\,X(t)\cos\omega_0 t$$

The ACF of o/p random process $Y(t) = R_{YY}(\tau)$

$$= A\left[E[Y(t)\ Y(t+\tau)]\right]$$

$$E[Y(t)\ Y(t+\tau)] = E\left[A\ X(t)\cos\omega_0 t \cdot A\ X(t+\tau)\cos(\omega_0 t+\tau)\right]$$

$$= E\left[A^2 X(t)\ X(t+\tau)\cos\omega_0 t\ \cos(\omega_0 t+\omega_0\tau)\right]$$

$$= A^2\ E[X(t)\ X(t+\tau)]\cos\omega_0 t\ \cos(\omega_0 t+\omega_0\tau)$$

We know $\Rightarrow$ $R_{XX}(\tau) = E[X(t)\ X(t+\tau)]$ for WSS

$$\cos A\cos B = \frac{\cos(A-B)+\cos(A+B)}{2}$$

$$= A^2 \cdot R_{XX}(\tau)\left[\frac{\cos(\omega_0 t-\omega_0 t-\omega_0\tau)+\cos(\omega_0 t+\omega_0 t+\omega_0\tau)}{2}\right]$$

$$= A^2 R_{XX}(\tau)\cdot\left[\frac{\cos(-\omega_0\tau)+\cos(2\omega_0 t+\omega_0\tau)}{2}\right]$$

$\because \cos(-\theta) = \cos\theta$

$$= A^2 R_{XX}(\tau)\left[\frac{\cos\omega_0\tau}{2}+\frac{\cos(2\omega_0 t+\omega_0\tau)}{2}\right]$$

$$= A^2 R_{XX}(\tau)\frac{\cos\omega_0\tau}{2} + \frac{A^2 R_{XX}(\tau)\cos(2\omega_0 t+\omega_0\tau)}{2}$$

$$E[X(t)X(t+\tau)] = \frac{A^2}{2}R_{XX}(\tau)\cos\omega_0\tau + \frac{A^2}{2}R_{XX}(\tau)\cos(2\omega_0 t+\omega_0\tau)$$

Now $R_{YY}(\tau) = A\left[E(Y(t)\ Y(t+\tau))\right]$

$$\& \ R_{YY}(\tau) = A\left[\frac{A^2}{2}R_{XX}(\tau)\cos(\omega_0\tau) + \frac{A^2}{2}R_{XX}(\tau)\cos(2\omega_0 t+\omega_0\tau)\right]$$

$$= A\left[\frac{A^2}{2}R_{XX}(\tau)\cos\omega_0\tau\right] + A\left[\frac{A^2}{2}R_{XX}(\tau)\cos(2\omega_0 t+\omega_0\tau)\right]$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\frac{A^2}{2}R_{XX}\cos(\omega_0\tau)\right]dt + \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left[\frac{A^2}{2}R_{XX}(\tau)\cos(2\omega_0 t+\omega_0\tau)\right]dt$$

$$= \frac{A^2}{2} R_{xx}(\tau) \cos(\omega_0 \tau) \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} 1\, dt + \frac{A^2}{2} R_{xx}(\tau) \underset{T\to\infty}{Lt} \frac{1}{2T}\int_{-T}^{T} \cos(2\omega_0 t + \omega_0 \tau)\, dt$$

$$= \frac{A^2}{2} R_{xx}(\tau) \cos \omega_0 \tau \underset{T\to\infty}{Lt} \frac{1}{2T} [t]_{-T}^{T} + \frac{A^2}{2} R_{xx}(\tau) \underset{T\to\infty}{Lt} \frac{1}{2T} (0)$$

$$\because \int_{-T}^{T} \cos(n\omega_0 t + \theta)\, dt = 0$$

$$= \frac{A^2}{2} R_{xx}(\tau) \cos(\omega_0 \tau) \underset{T\to\infty}{Lt} \frac{1}{2T} [T - (-T)] + 0$$

$$\boxed{\therefore R_{YY}(\tau) = \frac{A^2}{2} R_{xx}(\tau) \cos(\omega_0 \tau)}$$

Output PSD $= S_{YY}(\omega) = F[R_{YY}(\tau)]$

$$= F\left[\frac{A^2}{2} R_{xx}(\tau) \cos(\omega_0 \tau)\right]$$

$$\because F[a\, g(\tau)] = a\, F[g(\tau)]$$

$$= \frac{A^2}{2} F[R_{xx}(\tau) \cos(\omega_0 \tau)]$$

We know modulation theorem

$$g(\tau) \xleftrightarrow{F.T.} G(\omega) \text{ then}$$

$$g(\tau) \cos(\omega_0 \tau) \longleftrightarrow \frac{1}{2}[G(\omega - \omega_0) + G(\omega + \omega_0)]$$

$$R_{xx}(\tau) \longleftrightarrow S_{xx}(\omega)$$

$$R_{xx}(\tau) \cos(\omega_0 \tau) \longleftrightarrow \frac{1}{2}[S_{xx}(\omega - \omega_0) + S_{xx}(\omega + \omega_0)]$$

$$\Rightarrow S_{YY}(\omega) = \frac{A^2}{2} \times \frac{1}{2}[S_{xx}(\omega - \omega_0) + S_{xx}(\omega + \omega_0)]$$

$$\boxed{\therefore S_{YY}(\omega) = \frac{A^2}{4}[S_{xx}(\omega - \omega_0) + S_{xx}(\omega + \omega_0)]}$$

3. Find transfer function, impulse response of o/p PSD, o/p average power, o/p ACF, i/p ACF, and i/p average power for the RC low pass filter network when applied i/p having PSD is Gaussian White noise PSD i.e $\frac{N_0}{2}$ and also find noise bandwidth of RC low pass filter.

Sol: A RC low pass filter is as shown in figure.

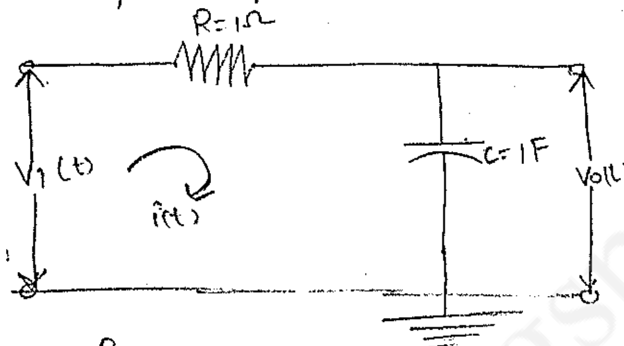


fig a: RC-Lowpass filter

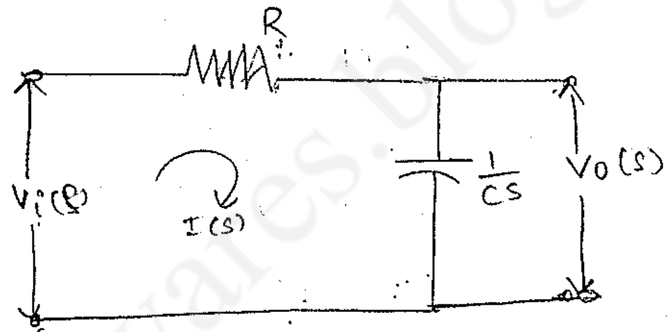


fig b: S-domain equivalent n/w

i/p voltage $= V_i(s) = R\cdot I(s) + \frac{1}{Cs} I(s) = \left(R + \frac{1}{Cs}\right) I(s)$

$$= \left[\frac{1 + RCS}{CS}\right] I(s)$$

o/p voltage $= V_o(s) = I(s) \cdot \frac{1}{CS}$

Transfer function in S-domain $= H(s) = \frac{V_o(s)}{V_i(s)}$

$$= \frac{\cancel{I(s)}\frac{1}{\cancel{C}S}}{\left[\frac{1 + RCS}{\cancel{C}S}\right] \cdot \cancel{I(s)}}$$

$$= \frac{1}{1 + RCS}$$

$$\therefore H(s) = \frac{1}{RC}\left[\frac{1}{\frac{1}{RC} + s}\right]$$

Transfer function in frequency domain (or) frequency response $= H(\omega)$

$$= H(j\omega) = H(s)\Big|_{j\omega = s}$$

$$= \frac{1}{RC}\left[\frac{1}{\frac{1}{RC} + j\omega}\right]$$

$$\therefore H(\omega) = \frac{1}{1 + jRC\omega}$$

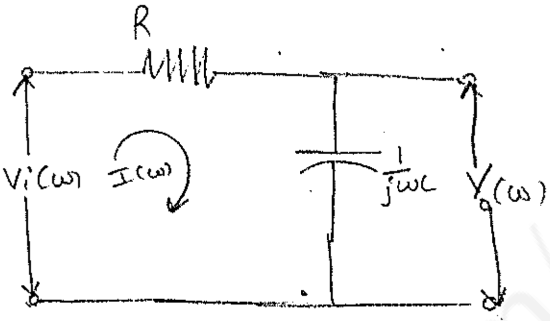


fig (c): frequency domain n/w

Magnitude Response $|H(\omega)| = \left|\frac{1}{1 + j\omega RC}\right|$

$$= \frac{1}{\sqrt{1 + (\omega RC)^2}}$$

$$= \frac{1}{(RC)}\left[\frac{1}{\sqrt{\left(\frac{1}{RC}\right)^2 + \omega^2}}\right]$$

$$|H(\omega)|^2 = \frac{1}{(RC)^2}\left[\frac{1}{\sqrt{\left(\frac{1}{RC}\right)^2 + \omega^2}}\right]$$

$$= \frac{1}{RC^2} \frac{1}{\left(\frac{1}{RC}\right)^2 + \omega^2}$$

$$|H(\omega)|^2 = \frac{1}{1 + \omega^2 R^2 C^2}$$

Phase response $= \angle H(\omega) = -\tan^{-1}\left(\frac{\omega RC}{1}\right)$

$$\angle H(\omega) = -\tan^{-1}(\omega RC) \left\{\because \frac{1}{a + jb} = -\tan^{-1}\left(\frac{b}{a}\right)\right\}$$

O/p PSD :

We know $S_{YY}(\omega) = |H(\omega)|^2 S_{XX}(\omega)$

Given i/p PSD $= S_{XX}(\omega) = \frac{N_0}{2}$ W/Hz

$$S_{YY}(\omega) = \frac{1}{1+\omega^2R^2C^2} \cdot \frac{N_0}{2}$$

$$\boxed{S_{YY}(\omega) = \frac{N_0}{2(1+\omega^2R^2C^2)} = \frac{N_0}{2R^2C^2((1/RC)^2+\omega^2)}}$$

O/P Average Power:

$$P_{YY} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{YY}(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{N_0}{2(1+\omega^2R^2C^2)}\, d\omega$$

$$= \frac{N_0}{2\pi \times 2}\int_{-\infty}^{\infty} \frac{1}{(1+\omega^2R^2C^2)}\, d\omega$$

$$= \frac{N_0}{4\pi}\int_{-\infty}^{\infty} \frac{1}{R^2C^2((1/RC)^2+\omega^2)}\, d\omega$$

$$= \frac{N_0}{4\pi R^2C^2}\int_{-\infty}^{\infty} \frac{1}{((1/RC)^2+\omega^2)}\, d\omega$$

$$= \frac{N_0}{4\pi \cancel{R^2C^2}} \cdot \frac{1}{(1/RC)^{\cancel{2}}} \tan^{-1}\left(\frac{\omega}{1/RC}\right)$$

$$= \frac{N_0}{4\pi}\left[\tan^{-1}(\omega RC)\right]_{-\infty}^{\infty}$$

$$= \frac{N_0}{4\pi}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right]$$

$$= \frac{N_0}{4\pi}\cdot[\pi]$$

$$\boxed{P_{YY} = \frac{N_0}{4RC}\ W/\cancel{Hz}}$$

O/p ACF:

$$R_{YY}(\tau) = F^{-1}\left[S_{YY}(\omega)\right]$$

$$= F^{-1}\left[\frac{N_0}{2R^2C^2\left(\frac{1}{(RC)^2}+\omega^2\right)}\right]$$

$$= \frac{N_0}{2R^2C^2} \; F^{-1}\left[\frac{1}{\left(\left(\frac{1}{RC}\right)^2+\omega^2\right)}\right]$$

We know $\quad e^{-a|\tau|} \xleftrightarrow{F.T} \dfrac{2a}{a^2+\omega^2}$

$$\frac{1}{2a}\, e^{-a|\tau|} \longleftrightarrow \frac{1}{a^2+\omega^2}$$

If $a = \frac{1}{RC} \Rightarrow \dfrac{1}{2\left(\frac{1}{RC}\right)}\, e^{-\frac{1}{RC}|\tau|} \longleftrightarrow \dfrac{1}{(1/RC)^2+\omega^2}$

$$F^{-1}\left[\frac{1}{\left(\frac{1}{RC}\right)^2+\omega^2}\right] = \frac{RC}{2}\, e^{-\frac{|\tau|}{RC}}$$

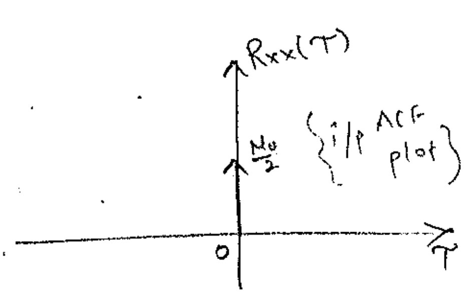
$$\therefore \; R_{YY}(\tau) = \frac{N_0}{2R^2C^2}\cdot\frac{RC}{2}\, e^{-\frac{1}{RC}|\tau|} = \frac{N_0}{4RC}\, e^{-\frac{1}{RC}|\tau|}$$

$$\boxed{R_{YY}(\tau) = \frac{N_0}{4RC}\, e^{-\frac{1}{RC}|\tau|}}$$

i/p ACF:

$$R_{XX}(\tau) = F^{-1}\left[S_{XX}(\omega)\right]$$

Given $\quad S_{XX}(\omega) = \dfrac{N_0}{2}$



$$R_{XX}(\tau) = F^{-1}\left[\frac{N_0}{2}\right]$$

$$= \frac{N_0}{2}\, F^{-1}[1]$$

We know $\delta(\tau) \longleftrightarrow 1 \Rightarrow F^{-1}[1] \leftrightarrow \delta(\tau)$

$$\boxed{R_{XX}(\tau) = \frac{N_0}{2}\,.\,\delta(\tau)} \text{ i.e. } \begin{cases} \frac{N_0}{2} & ; \; \tau = 0 \\ 0 & ; \; \tau \neq 0 \end{cases}$$

i/p Average Power:

$$P_{xx} = R_{xx}(0) = \frac{N_0}{2}\delta(0)$$

$$\boxed{\therefore P_{xx} = \frac{N_0}{2} \text{ Watts}}$$

Noise bandwidth:

The noise bandwidth of network is defined by

$$W_N = \frac{\int_0^\infty |H(\omega)|^2\, d\omega}{|H(0)|^2}$$

$$\int_0^\infty |H(\omega)|^2 d\omega = \int_0^\infty \frac{1}{(RC)^2}\left(\frac{1}{(\frac{1}{RC})^2 + \omega^2}\right) d\omega$$

$$= \frac{1}{(RC)^2}\left[\frac{1}{1/RC}\tan^{-1}\left(\frac{\omega}{(1/RC)}\right)\right]_0^\infty$$

$$\cancel{= \frac{RC}{R^2C^2}\tan^{-1}\left(\frac{\pi}{2} - 0\right)}$$

$$= \frac{RC}{R^2C^2}\left[\tan^{-1}\omega RC\right]_0^\infty$$

$$= \frac{1}{R.C}\left(\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi}{2RC}$$

$$|H(0)|^2 = \left(\frac{1}{RC}\right)^2\left[\frac{1}{(1/RC)^2 + 0^2}\right] = 1$$

$$\boxed{W_N = \frac{\pi/2RC}{1} = \pi/2RC \text{ rad/sec}}$$

Cutoff frequency $= W_c = \frac{1}{RC}$ rad/sec

$$\boxed{W_N = \frac{\pi}{2} W_c}$$

$$\boxed{B_N = \frac{\omega_N}{2\pi}}$$

$$\omega_N = 2\pi B_N \Rightarrow B_N = \frac{\omega_N}{2\pi}\ Hz$$

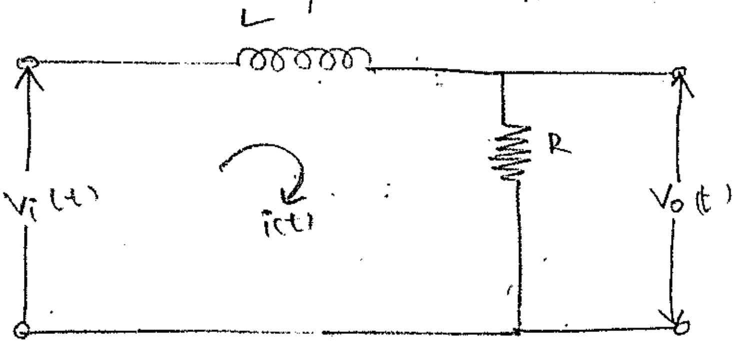
$$= \frac{\frac{\pi}{2RC}}{2\pi}$$

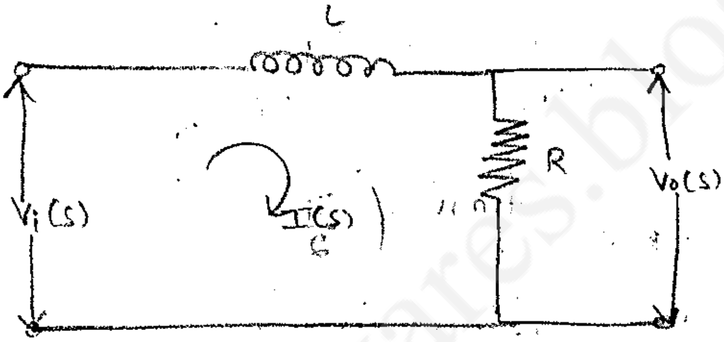
$$\boxed{B_N = \frac{\pi}{4\pi RC}\ Hz}$$

4. Repeat the above problem for the given RL n/w.

$H(S) = \frac{R}{R+SL}$; $H(j\omega) = \frac{R}{R+j\omega L}$

$h(t) = \frac{R}{L} e^{-\frac{R}{L}t} u(t)$





i/p voltage $= V_i(s) = SLI(s) + R[I(s)] = I(s)(R+L)$

O/p voltage $= V_o(s) = I(s) \cdot R = R\,I(s)$

Transfer function in S-domain $= H(s) = \frac{V_o(s)}{V_i(s)}$

$$= \frac{R\,(\cancel{I(s)})}{\cancel{I(s)}\,(R+j\omega L)}$$

$$\therefore H(s) = \frac{R}{R+j\omega L}$$

(OR)

$$H(s) = \frac{V_o(s)}{V_i(s)}$$

$$\boxed{\therefore H(s) = \frac{R}{R+SL}}$$

$$= \frac{\sqrt{\pi}}{2}\, G\left(\frac{\sqrt{\pi}\,\omega}{2}\right)$$

$$= \frac{\sqrt{\pi}}{2}\, e^{-\left(\frac{\left(\frac{\sqrt{\pi}\,\omega}{2}\right)^2}{4\pi}\right)} = \frac{\sqrt{\pi}}{2}\, e^{-\frac{\omega^2}{16}}$$

$$= \frac{\sqrt{\pi}}{2}\, e^{-\frac{\pi^2 f^2}{4}}$$

$$\therefore\; e^{-4\tau^2} \xleftrightarrow{F.T} \frac{\sqrt{\pi}}{2}\, e^{-\omega^2/16} = \frac{\sqrt{\pi}}{2}\, e^{-\frac{\pi^2 f^2}{4}}$$

$(\because \omega = 2\pi f$

$$\boxed{\therefore S_x(\omega) = 6\pi\delta(\omega) + 2\times\frac{\sqrt{\pi}}{2}\cdot e^{-\omega^2/16}}$$

(b) Average power in $X(t)$:

We know ACF of $X(t) = R_x(\tau) = 3 + 2e^{-4\tau^2}$

$$P_x = R_x(0) = 3 + 2e^{-4(0)^2} = 3 + 2 = 5$$

$$\boxed{P_x = 5\text{ W}}$$

(c)

$$P_x = \frac{1}{2\pi}\int_{-1/\sqrt{2}}^{1/\sqrt{2}} S_x(\omega)\, d\omega$$

$$= \frac{1}{2\pi}\int_{-1/\sqrt{2}}^{1/\sqrt{2}} \left(6\pi\delta(\omega) + \sqrt{\pi}\, e^{-\omega^2/16}\right) d\omega$$

$$= \frac{1}{2\pi}\times 6\pi \int_{-1/\sqrt{2}}^{1/\sqrt{2}} \delta(\omega)\, d\omega + \frac{1}{2\pi}\cdot\sqrt{\pi}\int_{-1/\sqrt{2}}^{1/\sqrt{2}} e^{-\frac{\omega^2}{16}}\, d\omega$$

$$= 3\times \delta(\omega)\Big|_{\omega=0} + \frac{1}{2\pi}\times\sqrt{\pi}\times 2\int_0^{1/\sqrt{2}} e^{-\frac{\omega^2}{16}}\, d\omega = 3\times 1 + \frac{1}{\sqrt{\pi}}\int_0 e^{-\left(\frac{\omega}{4}\right)^2} d\omega$$

Put $\frac{\omega}{4} = \tau \Rightarrow d\omega = 4\, d\tau$

$\omega \to 0 : \tau \to 0$

$\omega = 1/\sqrt{2} : \tau \to \frac{1}{4\sqrt{2}}$

$$= 3 + \frac{1}{\sqrt{\pi}} \int_0^{1/4\sqrt{2}} e^{-\tau^2} d\tau$$

$$= 3 + \frac{4}{\sqrt{\pi}} \phi\left(\frac{1}{4\sqrt{2}}\right) \qquad \phi(x) = \int_0^{x} e^{-\tau^2} d\tau.$$

$\tau \to \frac{1}{4\sqrt{2}} \to 0$